corner charges in gravity:

the effect of alternative boundary conditions

gloria odak

Aix-Marseille université

Centre de Physique Théorique

COST CA18108 Fourth Annual Conference - Rijeka

14/07/2023

intro: Bondi energy-loss formula

- one of the most useful equations of GR: $$M(u_1) - M(u_0) = -\frac{1}{8}\int_{u_0}^{u_1}\int_S News^2$$
- Considered historically as a *theoretical proof* that gravitational waves exist
- the first in a series of flux-balance laws that allow the reconstruction of a GW signal
- obtained from EEs via an asymptotic expansion far away from the source
- alternatively can be derived from an application of Noether's theorem to situations when the symplectic potential is not conserved in time
- this viewpoint has been the starting point for a series of recent developments, ranging from the phenomenology of GW to more theoretical questions of boundary symmetries in gauge theories and gravity

outline

- Part I: review of Noether's theorem and surface charges
- Part II: dependence of gravitational charges on boundary conditions
  - covariant phase space language
  - non-null boundaries
  - null boundaries
- Part III (time permitting): asymptotic symmetry algebras (extensions of BMS)

Noether's theorem

if the Lagrangian has a continuous symmetry, then there is a current which is conserved on-shell.

$$\delta_\epsilon L = dY_\epsilon \quad \Rightarrow \quad dj_\epsilon \approx 0, \quad j_\epsilon := I_\epsilon \theta - Y_\epsilon$$

$$j_\epsilon = C_\epsilon + dq_\epsilon$$

Noether's theorem

if the Lagrangian has a continuous symmetry, then there is a current which is conserved on-shell.

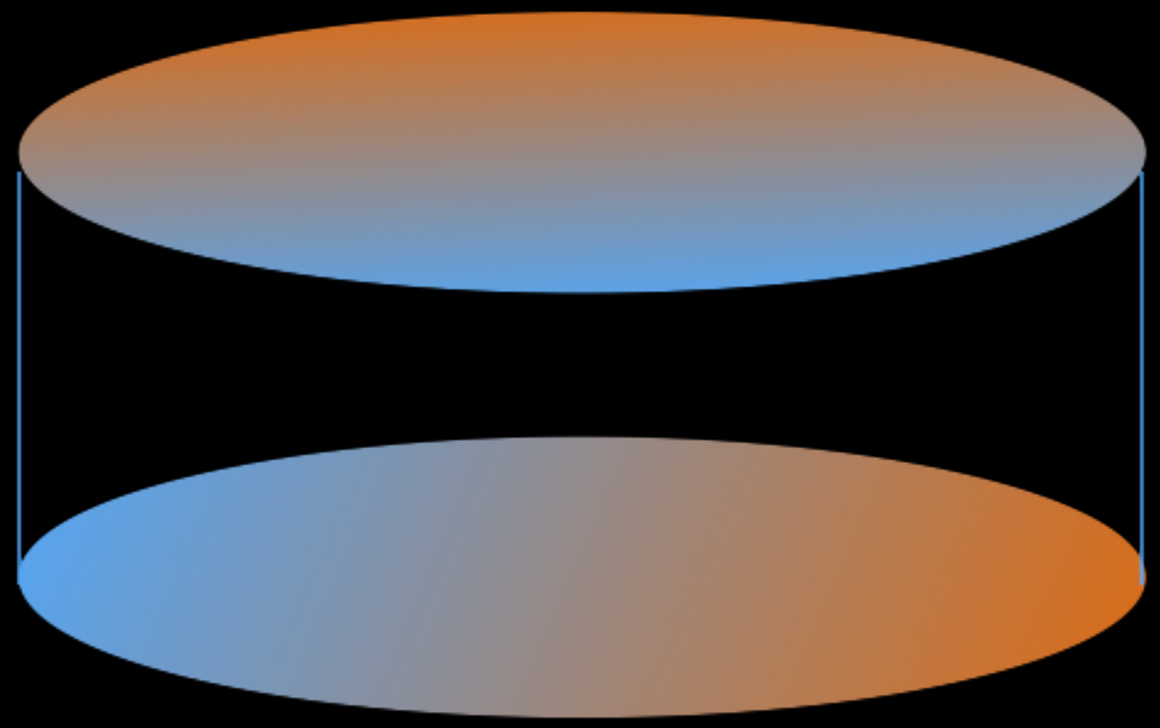
$$\delta_\epsilon L = dY_\epsilon \quad \Rightarrow \quad dj_\epsilon \approx 0, \quad j_\epsilon := I_\epsilon \theta - Y_\epsilon$$

$$j_\epsilon = C_\epsilon + dq_\epsilon$$

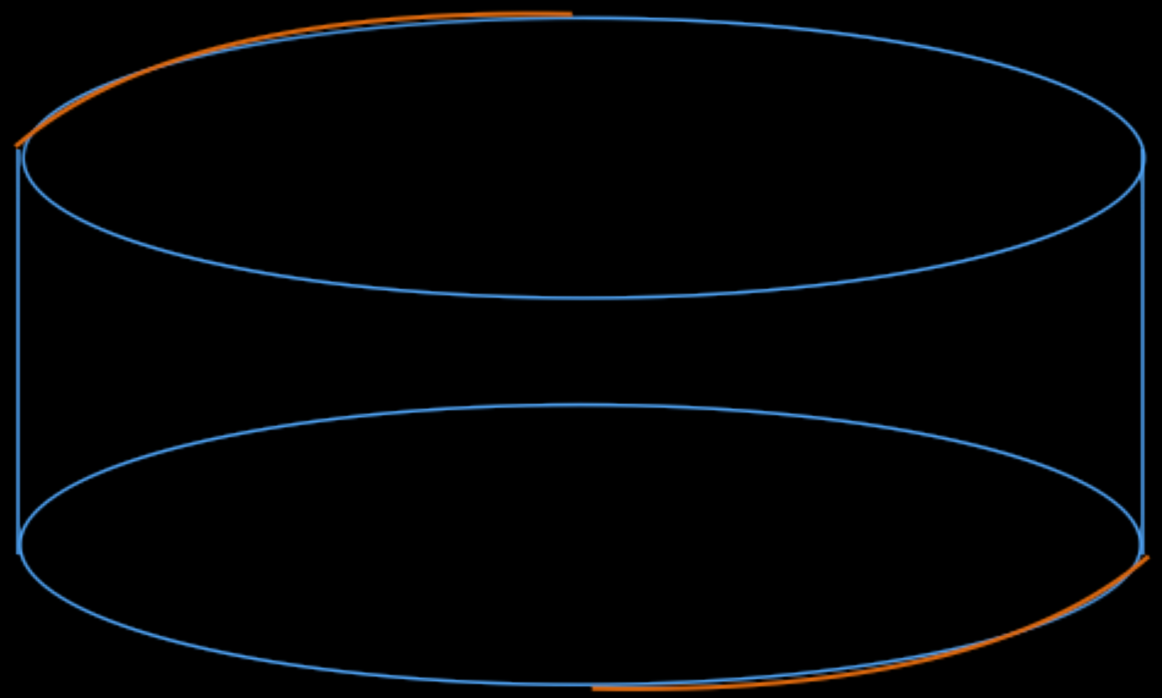
= constraint (global charge) + boundary term (surface charge)

Noether's theorem

Textbook examples:
✳Global U(1) invariance: conservation of electric charge
✳Poincaré invariance: conserved energy-momentum tensor



Less known:
✳local U(1) invariance: conservation of surface charges
✳diffeo invariance: conservation of surface charges



Noether's theorem

Einstein-Hilbert: $L^{EH} = R\epsilon$

infinitesimal continuous symmetry: $\delta_\xi g_{\mu\nu} = \pounds_\xi g_{\mu\nu} \Rightarrow \delta_\xi L = d i_\xi L$

Nother current: $j_\xi = I_\xi \theta - Y_\xi = i_\xi E + d q_\xi$

$$j_\xi = G^\mu_\nu \xi^\nu - \nabla_\nu \nabla^{[\mu} \xi^{\nu]}$$

$$q_\xi = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \nabla^\rho \xi^\sigma dx^\mu \wedge dx^\nu$$

Noether's theorem

Einstein-Hilbert: $$L^{EH} = R\epsilon$$

infinitesimal continuous symmetry: $$\delta_\xi g_{\mu\nu} = \pounds_\xi g_{\mu\nu} \Rightarrow \delta_\xi L = d i_\xi L$$

Nother current: $$j_\xi = I_\xi \theta - Y_\xi = i_\xi E + d q_\xi$$

$$j_\xi = G^\mu_\nu \xi^\nu - \nabla_\nu \nabla^{[\mu} \xi^{\nu]}$$

$$q_\xi = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} \nabla^\rho \xi^\sigma dx^\mu \wedge dx^\nu$$

for Kerr:

$$\int_S q_{\partial_\varphi} = aM$$

$$\int_S q_{\partial_t} = \frac{M}{2}$$

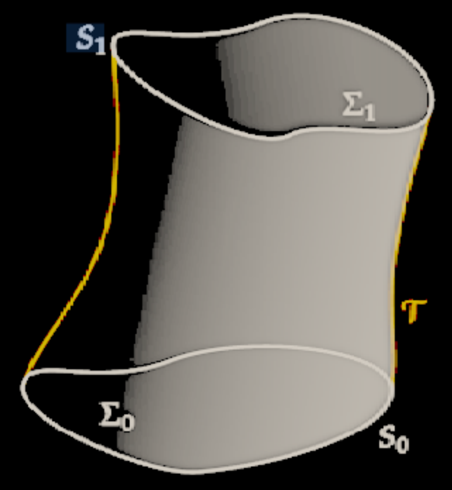
the role of boundary conditions

Gravitational radiation makes the definition of energy ambiguous

the role of boundary conditions

Gravitational radiation makes the definition of energy ambiguous

This ambiguity is commonly fixed by imposing Dirichlet b.c.



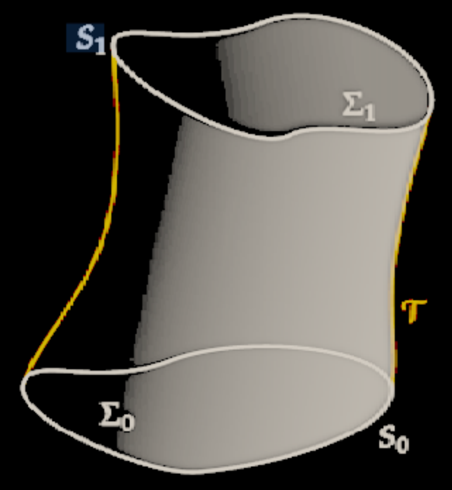
Asymptotically flat — ADM

Quasilocal — BY

the role of boundary conditions

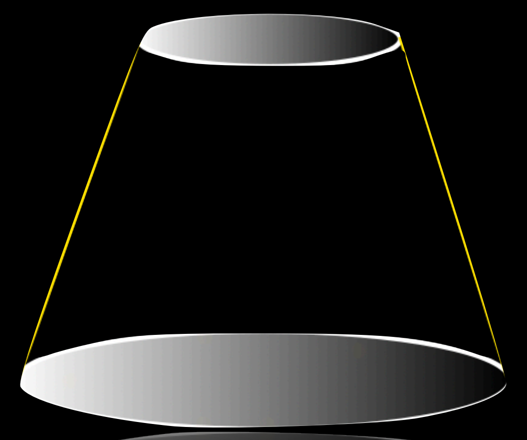
Gravitational radiation makes the definition of energy ambiguous

This ambiguity is commonly fixed by imposing Dirichlet b.c.



Asymptotically flat — ADM, BMS

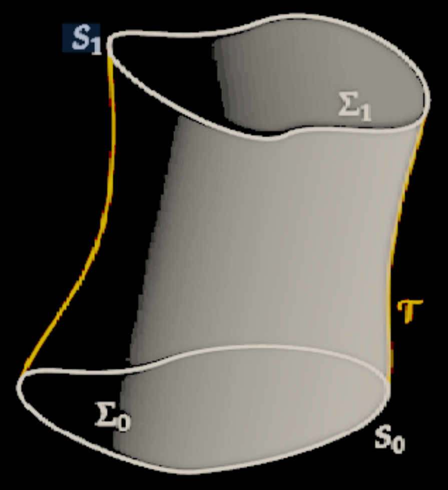
Quasilocal — BY (up to anomalies)



the role of boundary conditions

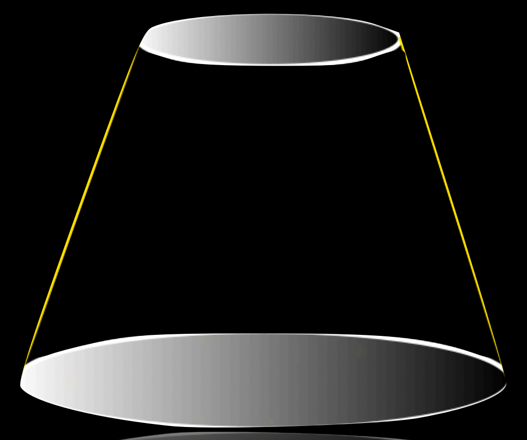
Gravitational radiation makes the definition of energy ambiguous

This ambiguity is commonly fixed by imposing Dirichlet b.c.



Asymptotically flat — ADM, BMS

Quasilocal — BY (up to anomalies)



Analogy with thermodynamics:

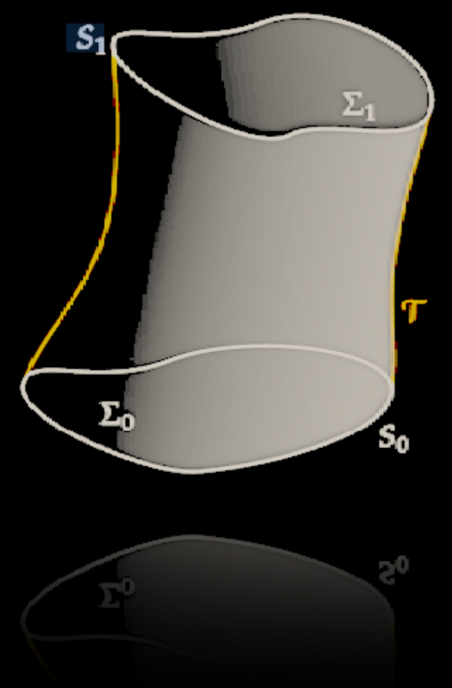
Dirichlet = isothermal — internal energy

Neumann = adiabatic — free energy

the role of boundary conditions

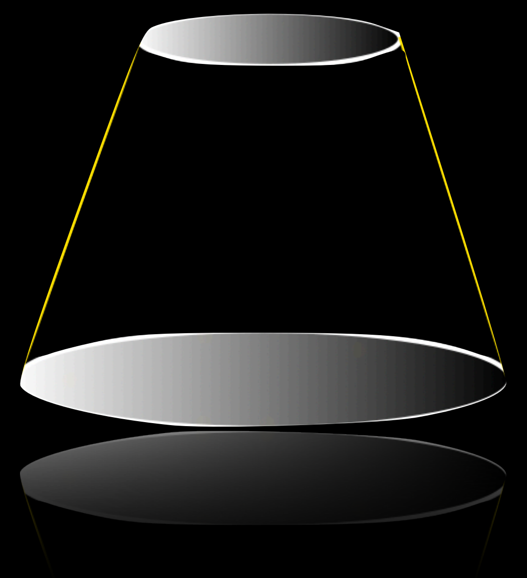
Gravitational radiation makes the definition of energy ambiguous

This ambiguity is commonly fixed by imposing Dirichlet b.c.



Asymptotically flat — ADM, BMS

Quasilocal — BY (up to anomalies)



Analogy with thermodynamics:

Dirichlet = isothermal — internal energy

Neumann = adiabatic — free energy

How about gravity? What happens to energy if we use different types of boundary conditions?

covariant phase space

[Iyer, Wald '94]
[Wald, Zupas '00]

Start with the Einstein-Hilbert Lagrangian

$$L^{EH} = R\epsilon$$

covariant phase space

[Iyer, Wald '94]

[Wald, Zupas '00]

Start with the Einstein-Hilbert Lagrangian

$$L^{EH} = R\epsilon$$

Arbitrary variation gives $\delta L^{EH} = G_{\mu\nu}\delta g^{\mu\nu}\epsilon + d\theta^{EH}$

covariant phase space

[Iyer, Wald '94]
[Wald, Zupas '00]

Start with the Einstein-Hilbert Lagrangian

$$L^{EH} = R\epsilon$$

Arbitrary variation gives $\delta L^{EH} = G_{\mu\nu}\delta g^{\mu\nu}\epsilon + d\theta^{EH}$

$$\theta^{\mathrm{EH}} = s\left(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K\right)\epsilon_\Sigma + d\vartheta \qquad \delta L^{EH} \not\approx 0$$

covariant phase space

[Iyer, Wald '94]
[Wald, Zupas '00]

Start with the Einstein-Hilbert Lagrangian

$$L^{EH} = R\epsilon$$

Arbitrary variation gives $\delta L^{EH} = G_{\mu\nu}\delta g^{\mu\nu}\epsilon + d\theta^{EH}$

$$\theta^{\mathrm{EH}} = s\left(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K\right)\epsilon_{\Sigma} + d\vartheta \qquad \delta L^{EH} \not\approx 0$$

Need boundary Lagrangian $L = L^{EH} + d\ell$

covariant phase space

[Iyer, Wald '94]
[Wald, Zupas '00]

Start with the Einstein-Hilbert Lagrangian

$$L^{EH} = R\epsilon$$

Arbitrary variation gives $\delta L^{EH} = G_{\mu\nu}\delta g^{\mu\nu}\epsilon + d\theta^{EH}$

$$\theta^{\mathrm{EH}} = s\left(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K\right)\epsilon_\Sigma + d\vartheta \qquad \delta L^{EH} \not\approx 0$$

Need boundary Lagrangian $L = L^{EH} + d\ell$

Now $\theta := \theta^{EH} + \delta\ell - d\vartheta \qquad \theta \approx 0$

covariant phase space

[Iyer, Wald '94]
[Wald, Zupas '00]

Start with the Einstein-Hilbert Lagrangian

$$L^{EH} = R\epsilon$$

Arbitrary variation gives $\delta L^{EH} = G_{\mu\nu}\delta g^{\mu\nu}\epsilon + d\theta^{EH}$

$$\theta^{\mathrm{EH}} = s\left(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K\right)\epsilon_\Sigma + d\vartheta \qquad \delta L^{EH} \not\approx 0$$

Need boundary Lagrangian $L = L^{EH} + d\ell$

Now $\theta := \theta^{EH} + \delta\ell - d\vartheta \qquad \theta \approx 0$

[Harlow, Wu, '18]
[Freidel, Geiller, Pranzetti, '20]
[GO, Rignon-Bret, Speziale '22]

Improved Noether charge prescription

$$\delta H_\xi = -\int_\Sigma I_\xi \delta\theta \qquad H_\xi = \int_S q_\xi + i_\xi \ell - I_\xi \vartheta$$

boundary conditions and phase space polarization

Symplectic potential $\theta := \theta^{EH} + \delta\ell - d\vartheta$ in its most basic form is given by *pdq*

Boundary conditions in their most basic form are given by *dq=0*

DIRICHLET: $\delta q = 0$
NEUMAN: $\delta p = 0$
MIXED: $a\delta q + b\delta p = 0$

↔

DIFFERENT POLARIZATIONS OF THE PHASE SPACE

($\delta q = 0$ FOR DIFFERENT DEF OF q)

pullback of θ^{EH} to the timelike boundary

$$\theta^{\rm EH} = s\left(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K\right)\epsilon_\Sigma + d\vartheta^{\rm EH} = \tilde{\Pi}_{\mu\nu}\delta q^{\mu\nu}\epsilon_T + \delta(2K\epsilon_T) + d\vartheta^{EH}$$

DIRICHLET BOUNDARY CONDITIONS: $\delta q_{\mu\nu} = 0$ → $\ell = 2K\epsilon_T$

pullback of θ^{EH} to the timelike boundary

$$\theta^{\mathrm{EH}} = s\left(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K\right)\epsilon_\Sigma + d\vartheta^{\mathrm{EH}} = \tilde{\Pi}_{\mu\nu}\delta q^{\mu\nu}\epsilon_T + \delta(2K\epsilon_T) + d\vartheta^{EH}$$

$$\theta^{\mathrm{EH}} = -\,q_{\mu\nu}\delta\tilde{\Pi}^{\mu\nu}d^3x + d\vartheta^{\mathrm{EH}}$$

pullback of θ^{EH} to the timelike boundary

NEUMANN BOUNDARY CONDITIONS: $\delta\tilde{\Pi}^{\mu\nu} = 0$ → $\ell = 0$

$$\theta^{\text{EH}} = -\, q_{\mu\nu}\delta\tilde{\Pi}^{\mu\nu}d^3x + d\vartheta^{\text{EH}}$$

pullback of θ^{EH} to the timelike boundary

One particular choice of mixed bc that is geometrically motivated: [York '86]

Fixed conformal induced metric: $\hat{q}_{\mu\nu} := q^{-1/3} q_{\mu\nu}$ and the trace of extrinsic curvature $\delta \hat{q}_{\mu\nu} = 0 = \delta K$

pullback of θ^{EH} to the timelike boundary

One particular choice of mixed bc that is geometrically motivated: [York '86]

Fixed conformal induced metric: $\hat{q}_{\mu\nu} := q^{-1/3} q_{\mu\nu}$ and the trace of extrinsic curvature $\delta\hat{q}_{\mu\nu} = 0 = \delta K$

$$\theta^{\mathrm{EH}} = s\left(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K\right)\epsilon_{\Sigma} + d\vartheta^{\mathrm{EH}} = -\,(P^{\mu\nu}\delta\hat{q}_{\mu\nu} + \frac{4}{3}\delta K)\epsilon_T - \frac{2}{3}\delta(K\delta q) + d\vartheta^{EH}$$

pullback of θ^{EH} to the timelike boundary

One particular choice of mixed bc that is geometrically motivated: [York '86]

Fixed conformal induced metric: $\hat{q}_{\mu\nu} := q^{-1/3} q_{\mu\nu}$ and the trace of extrinsic curvature $\delta\hat{q}_{\mu\nu} = 0 = \delta K$

$$\theta^{\text{EH}} = -\,(P^{\mu\nu}\delta\hat{q}_{\mu\nu} + \frac{4}{3}\delta K)\epsilon_T - \frac{2}{3}\delta(K\delta q) + d\vartheta^{EH}$$

YORK BOUNDARY CONDITIONS: $\delta\hat{q}_{\mu\nu} = 0 \;\&\; \delta K = 0$ → $\ell = \frac{2}{3}K\epsilon_T$

All these cases can be parametrized by a real parameter b

$$\ell^b = bK\epsilon_\Sigma$$

$$L = L^{EH} + d\ell^b$$

$$\theta := \theta^{EH} + \delta\ell^b - d\vartheta^{EH}$$

(always with the same corner symplectic potential)

$$\vartheta^{\mathrm{EH}} := -u_\mu \delta n^\mu \epsilon_S = u^\mu n^\nu \delta g_{\mu\nu} \epsilon_S$$

boundary conditions	*quantity fixed on boundary*	*value of b*
Dirichlet	$q_{\mu\nu}$	2
York	$(\hat{q}_{\mu\nu}, K)$	2/3
Neumann	$\tilde{\Pi}^{\mu\nu}$	0

$$H^b_\xi = \int_S q^{\mathrm{EH}}_\xi + i_\xi \ell^b - I_\xi \vartheta^{EH} = \dots$$

Dirichlet (b=2) $H_\xi^{\text{BY}} = -2\int_S n^\mu \xi^\nu \left(\bar{K}_{\mu\nu} - \bar{q}_{\mu\nu}\bar{K}\right)\epsilon_S = -2\int_S n^\mu \xi^\nu \bar{\Pi}_{\mu\nu}\epsilon_S$ [Brown, York '93]

York (b=2/3) $H_\xi^{\text{Y}} = -2\int_S n^\mu \xi^\nu \left(\bar{K}_{\mu\nu} - \frac{1}{3}\bar{q}_{\mu\nu}\bar{K}\right)\epsilon_S = -2\int_S n^\mu \xi^\nu \bar{\Pi}_{\langle\mu\nu\rangle}\epsilon_S$

Neumann (b=0) $H_\xi^{\text{N}} = -2\int_S n^\mu \xi^\nu \bar{K}_{\mu\nu}\epsilon_S = -2\int_S n^\mu \xi^\nu \left(\bar{\Pi}_{\mu\nu} - \frac{1}{2}\bar{q}_{\mu\nu}\bar{\Pi}\right)\epsilon_S$

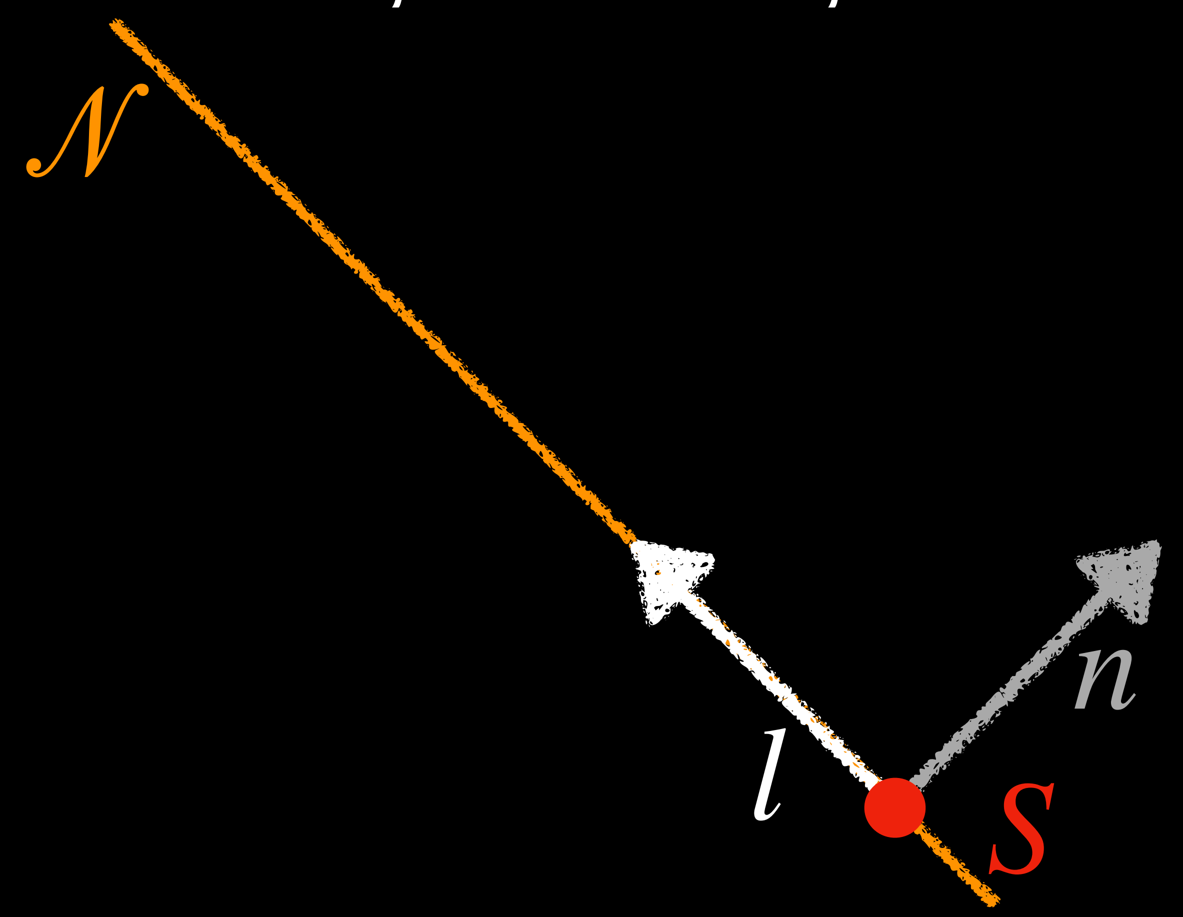
boundary conditions	*quantity held fixed*	*value of b*	*quasi-local energy*	*Kerr (renormalized)*
Dirichlet	$q_{\mu\nu}$	2	k	M
York	$(\hat{q}_{\mu\nu}, K)$	2/3	$k - 2\bar{K}/3$	$2M/3$
Neumann	$\tilde{\Pi}^{\mu\nu}$	0	$k - \bar{K}$	$M/2$

[GO, Speziale '21]

null boundary

$\Phi = 0 \to l_\mu = -f\partial_\mu \Phi$ $\quad l^2 = 0 \quad l^\mu \nabla_\mu l_\nu = k l_\nu$

$n^2 = 0, \quad l \cdot n = -1$



$$\gamma_{\mu\nu} = g_{\mu\nu} + 2 l_{(\mu} n_{\nu)}$$

$$B_{\mu\nu} := \gamma_\mu^\rho \gamma_\nu^\sigma \nabla_\rho l_\sigma = \sigma_{\mu\nu} + \frac{1}{2}\theta\gamma_{\mu\nu}$$

pullback of θ^{EH} to the null boundary

$$\underset{\leftarrow}{\theta}^{\mathrm{EH}} = \int_{\mathcal{N}} \left[B^{\mu\nu}\delta\gamma_{\mu\nu} - 2\omega_\mu \delta l^\mu + 2\delta(\theta + k) + \partial_n l^2 n^\mu \delta l_\mu \right] \epsilon_{\mathcal{N}} + d\vartheta^{\mathrm{EH}}$$

DIRICHLET BOUNDARY CONDITIONS: $\delta\gamma_{\mu\nu} = 0 \;\&\; \delta l^\mu = 0$ → $\ell = 2(k+\theta)\epsilon_{\mathcal{N}}$

[Parattu, Chakraborty, Majhi,Padmanabhan, '16]

pullback of θ^{EH} to the null boundary

$$\underset{\leftarrow}{\theta}^{\text{EH}} = \int_{\mathcal{N}} \left[B^{\mu\nu}\delta\gamma_{\mu\nu} - 2\omega_\mu \delta l^\mu + 2\delta(\theta + k) + \partial_n l^2 n^\mu \delta l_\mu \right] \epsilon_{\mathcal{N}} + d\vartheta^{\text{EH}}$$

DIRICHLET BOUNDARY CONDITIONS: $\delta\gamma_{\mu\nu} = 0 \;\&\; \delta l^\mu = 0$ → $\ell = 2(k + \theta)\epsilon_{\mathcal{N}}$

[Parattu, Chakraborty, Majhi,Padmanabhan, '16]

[Lehner, Myers, Poisson, Sorkin'16]

$$\ell^{ct} = -2\theta \ln \theta \epsilon_{\mathcal{N}}$$

pullback of θ^{EH} to the null boundary

$$\underset{\leftarrow}{\theta}^{\mathrm{EH}} = \int_{\mathcal{N}} \left[B^{\mu\nu}\delta\gamma_{\mu\nu} - 2\omega_\mu \delta l^\mu + 2\delta(\theta + k) + \partial_n l^2 n^\mu \delta l_\mu \right] \epsilon_{\mathcal{N}} + d\vartheta^{\mathrm{EH}}$$

$$= \int_{\mathcal{N}} \left[\sigma^{\mu\nu}\delta\hat{\gamma}_{\mu\nu} + \delta\left(2\theta + k\right) + 2\omega_\mu \delta l^\mu \right] \epsilon_{\mathcal{N}} + 2\delta\left(\theta\epsilon_{\mathcal{N}}\right) + d\vartheta^{EH}$$

“YORK” BOUNDARY CONDITIONS: $\delta\hat{\gamma}_{\mu\nu} = 0 \;\&\; \delta(k + 2\theta) = 0 \;\&\; \delta l^\mu = 0$ → $\ell = 2\theta\epsilon_{\mathcal{N}}$

properties of the charges

- canonical charges by Chandrasekaran, Flanagan, Prabhu ’18 conserved on non-expanding horizons defined with Dirichlet polarization
- changing to York polarization leads to charges conserved both on NEH and on minkowski lightcones
- might have interesting implications for dynamical processes

properties of the charges

- canonical charges by Chandrasekaran, Flanagan, Prabhu ’18 conserved on non-expanding horizons defined with Dirichlet polarization
- changing to York polarization leads to charges conserved both on NEH and on minkowski lightcones
- might have interesting implications for dynamical processes

[Wald (wip)]
[Rignon-Bret ‘23]

Alternative boundary conditions on null hypersurfaces

Gloria Odak, Antoine Rignon-Bret and Simone Speziale
Aix Marseille Univ., Univ. de Toulon, CNRS, CPT, UMR 7332, 13288 Marseille, France

2307.XXXX

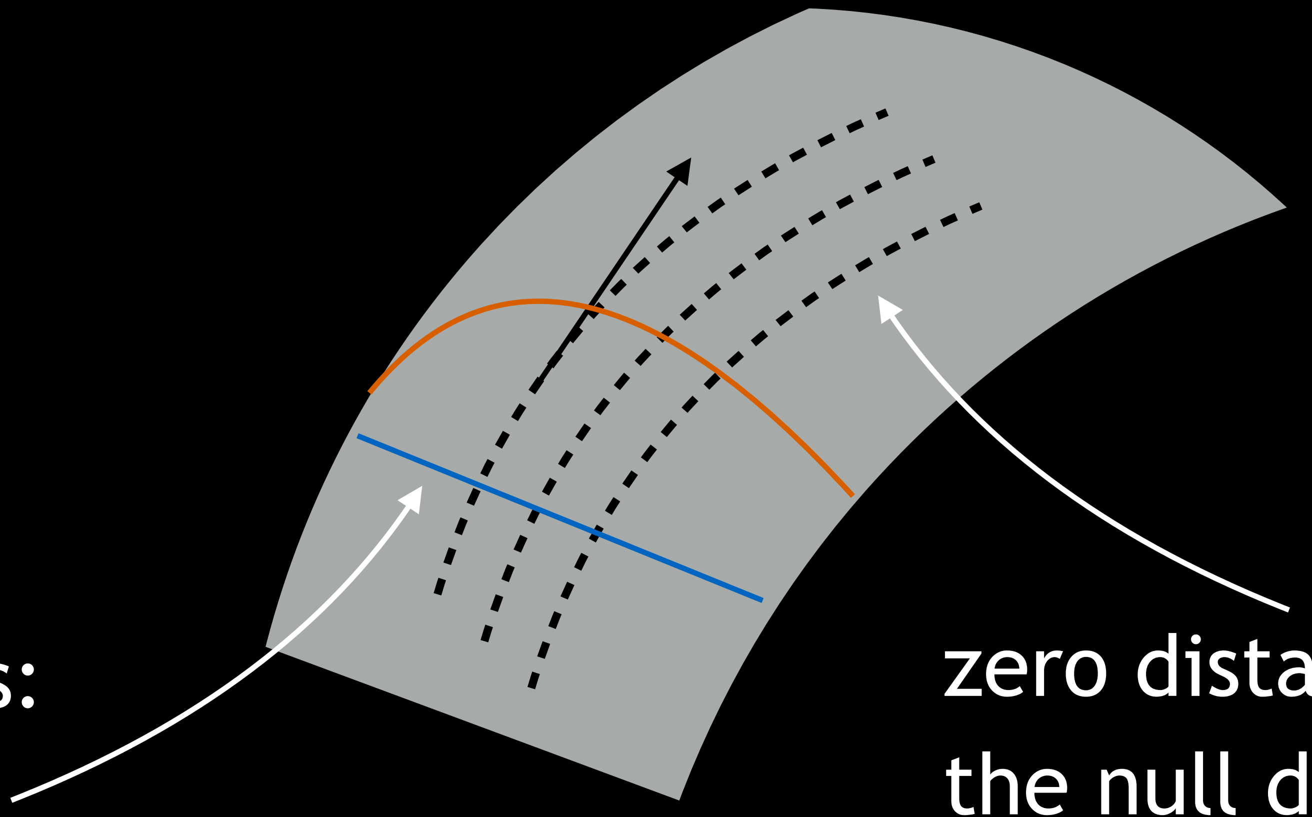
Why an infinite-dimensional symmetry group?

because scri is null, there is a degenerate direction and the inertial observers are not mapped among themselves by the Poncaré group but by an infinite-dimensional extension

The big difference is that a null hyperplane has a degenerate metric:
there is no distinguished notion of Cartesian coordinate in the degenerate direction

⇨ freedom of non-rigid translations: **supertranslations**

$$BMS = SL(2,\mathbb{C}) \ltimes \mathbb{R}$$



transverse directions:
distances measured
by the round 2-sphere metric

zero distances along
the null directions

the blue cut and the orange cut are equivalently good observers

why look for larger symmetries?

- subleading soft theorems — generalized BMS [Strominger group, Campiglia-Larddha,…] [Compere, Fiorucci, Ruzziconi '18]
- holography — extended BMS [Barnich, Trossaert '11]
- larger symmetry = more control over quantization [Barnich, Grumiller, Freidel and many more]
- the algebra of quasi-local observables on null surfaces is larger than BMS [Donnay, Giribet, Gonzalez, Pino '16] [Chandrasekara, Flannagan,Prabhu '18] [Freidel, Oliveri, Pranzetti, Speziale '21] …
- information loss paradox [Hawking, Strominger, Perry '16]

corner charge algebras

- at $\mathscr{I}$: BMSW algebra

$$\textbf{diff (S)} \ltimes \mathbb{R} \ltimes \mathbb{R}$$

[Freidel, Oliveri, Pranzetti, Speziale '21]

- at finite distance: extended corner symmetry algebra

[Donnelly, Freidel '16]

[Freidel, Oliveri, Pranzetti, Speziale '21]

$$\textbf{diff (S)} \ltimes \textbf{sl}(2,\mathbb{R}) \ltimes \mathbb{R}$$

- generic covariant theory: universal symmetry algebra

[Ciambelli, Leigh '21,'22]

$$\textbf{diff (S)} \ltimes \textbf{gl}(2,\mathbb{R}) \ltimes \mathbb{R}$$