Finsler spacetimes with (α,β)-metrics and their isometries

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Talk based on:

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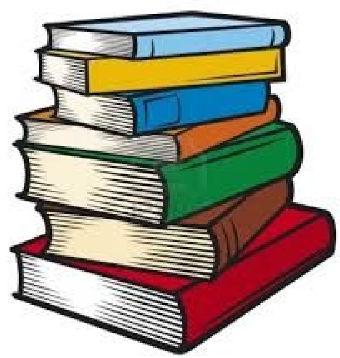


1 Main results

★ (α, β)-**metrics:**

- most commonly used class of Finsler metrics
- obtained by deforming a Riemannian metric (α) by a 1-form (β)

For general Finsler metrics of (α, β)-type:

✓ We find a set of (minimal) conditions such that these admit a *well defined causal structure*.

✓ We determine all Killing vector fields...

2 Motivation

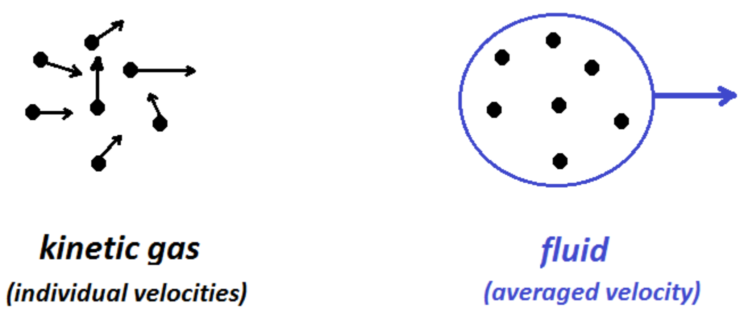
Modified dispersion relations −> Finslerian arc length

Also:

✓ SME models (Kostelecky&co.);

✓ minimal extension of Ehlers-Pirani-Schild axioms (Perlick&Laemmerzahl);

✓ more accurate description of gravity, takings into account *multiple sources, with different velocities* - **kinetic gas** (Hohmann, Pfeifer, NV, 2020)



3 Finsler gravity

General relativity $g_{\mu\nu}(x)$	Finsler gravity $L(x, \dot{x})$
$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$	$ds^2 = L(x, dx)$
$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\nu\sigma}(x)\frac{d\dot{x}^\nu}{ds}\frac{d\dot{x}^\sigma}{ds} = 0$	$\frac{d^2x^\mu}{ds^2} + N^\mu_\nu\left(x, \frac{d\dot{x}}{ds}\right)\frac{d\dot{x}^\nu}{ds} = 0$
$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}$	$\frac{1}{2}g^{\mu\nu}\frac{\partial^2 Ric}{\partial\dot{x}^\mu\partial\dot{x}^\nu} - 3L^{-1}Ric - \mathrm{div}\, P = \kappa\phi$

✓ Finslerian Ricci scalar Ric:

$$Ric := R_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{1}$$

Take: M - 4-dim manifold.

Finsler spacetime structure on M: a smooth scalar function

$$L = L\left(x, \dot{x}\right) \tag{2}$$

on some subset of TM, with:

(i) 2-homogeneity: $L(x, \alpha\dot{x}) = \alpha^2 L(x, \dot{x})$, $\forall \alpha > 0$;

(ii) Future timelike cones $\mathcal{T}_x$

at each $x \in M$, $\exists$ a connected conic set $\mathcal{T}_x \neq \emptyset$ such that:

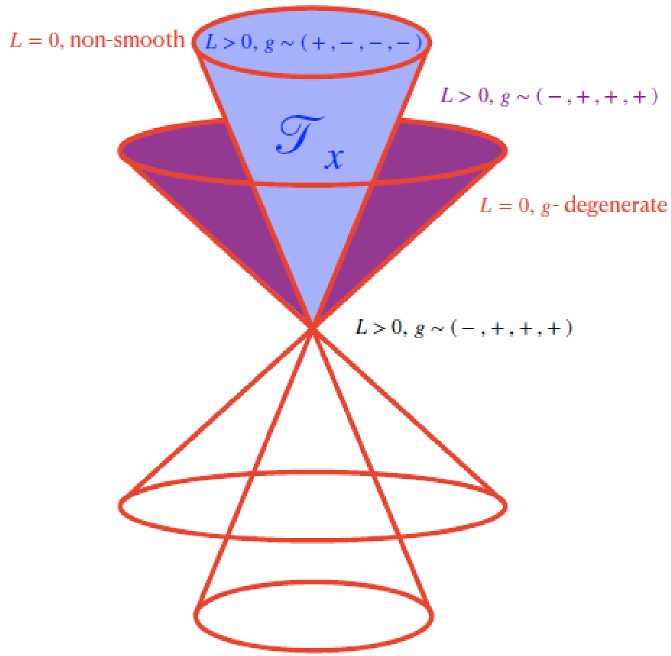
✓ $L > 0$ on $\mathcal{T}_x$, $L_{|\partial \mathcal{T}_x} = 0$

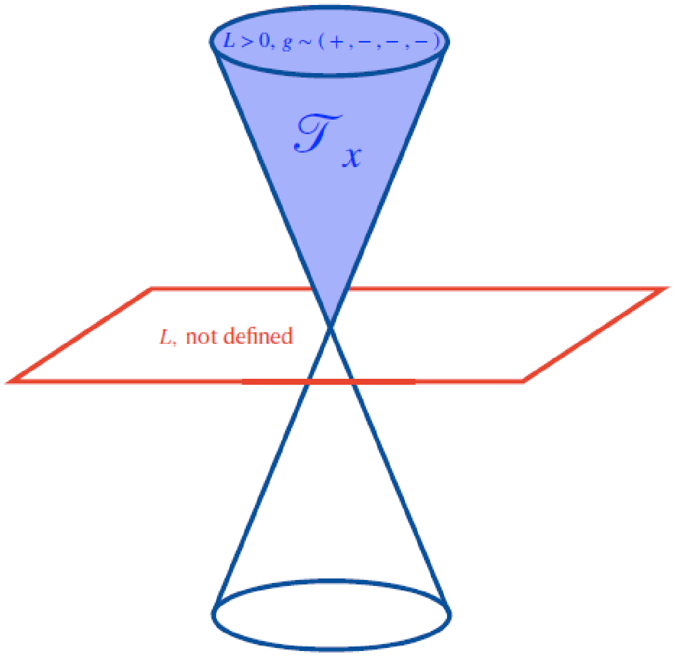
✓ The matrix

$$g_{\mu\nu}(x, \dot{x}) = \frac{1}{2}\frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu}$$

has Lorentzian signature $(+, -, -, -)$.

Finslerian cones:





4 (α, β)-metric spacetimes

Setting: $(M, a_{\mu\nu}(x))$ - Lorentzian (GR) spacetime, $b := b_\mu(x)\, dx^\mu$.

$$A(x, \dot{x}) \; : \; = a_{\mu\nu}(x)\, \dot{x}^\mu \dot{x}^\nu, \tag{3}$$

$$B(x, \dot{x}) \; : \; = b_\mu(x)\, \dot{x}^\mu. \tag{4}$$

Definition: $L = L(x, \dot{x})$ is an (α, β)-**metric** if:

$$L = A\Psi(s), \quad s = \frac{B^2}{A}. \tag{5}$$

Traditional notations:

$$|A| = \alpha^2, \; B = \beta \tag{6}$$

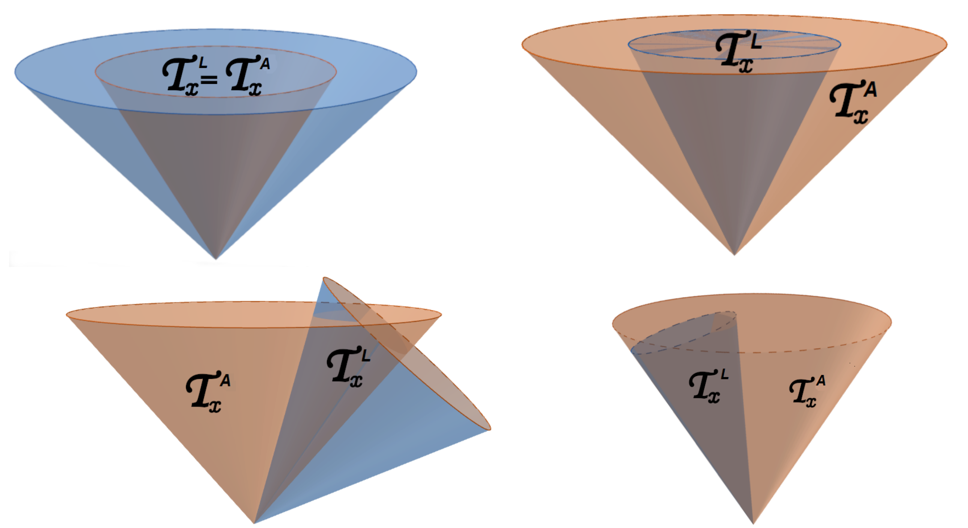
Assumption:

$$\mathcal{T}_x^A \cap \mathcal{T}_x^L \neq \varnothing. \tag{7}$$

Then:

$$\mathcal{T}_x^L \quad : \quad A > 0, \quad \Psi > 0. \tag{8}$$

$$\partial \mathcal{T}_x^L \quad : \quad A = 0 \quad or \quad \Psi = 0. \tag{9}$$



Determinant of $g_{\mu\nu}(x, \dot{x})$:

$$\det(g_{\mu\nu}) = \Psi^2(\Psi - s\Psi')\det(a)\frac{d}{ds}\left[(s - \langle b, b\rangle)\boldsymbol{\sigma}\right], \tag{10}$$

where:

$$\boldsymbol{\sigma} := \frac{(\Psi - s\Psi')^2}{\Psi} \tag{11}$$

Theorem 1: $L = A\Psi(s)$ - Finsler spacetime structure $\Leftrightarrow$ at each $x \in M$, $\exists$ a connected conic set $\mathcal{T}_x$ such that:

(i) $A > 0$, $\Psi > 0$ on $\mathcal{T}_x$ and $\lim\limits_{\dot{x} \to \partial\mathcal{T}_x} (A\Psi) = 0$.

(ii) at all $\dot{x} \in \mathcal{T}_x$:

$$\Psi - s\Psi' > 0, \quad (s - \langle b, b\rangle)\frac{d}{ds}\ln\boldsymbol{\sigma} > -1. \tag{12}$$

5 Examples

1. Randers metrics

$$L = \epsilon(\sqrt{|A|} + B)^2 = A\left(1 - \sqrt{s}\right)^2. \tag{13}$$

(where $\epsilon = sgn(A)$)

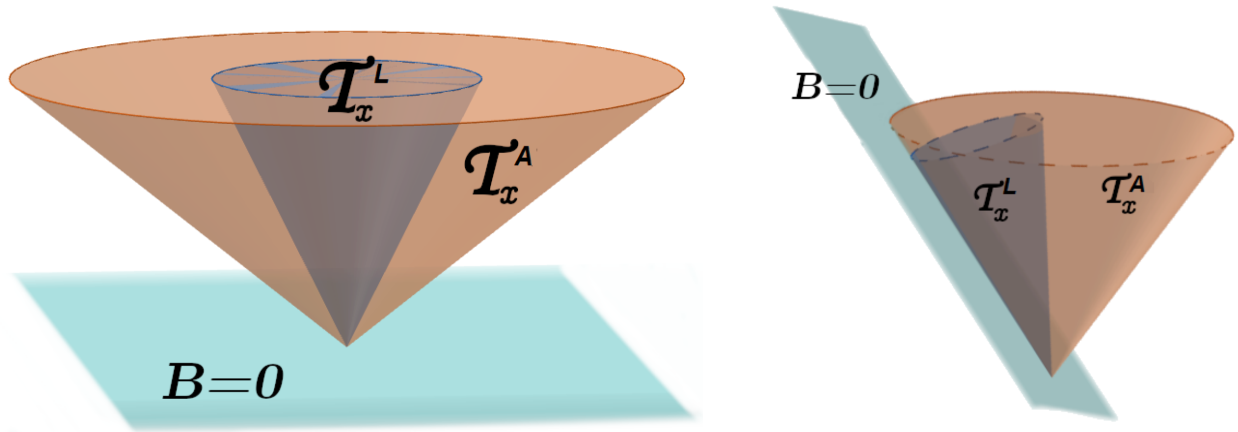
Proposition 1: L defines a Finsler spacetime structure $\Leftrightarrow$

$$\langle b, b\rangle \in [0, 1). \tag{14}$$

In this case:

$$\mathcal{T}_x^L = \{\dot{x} \mid (a_{\mu\nu} - b_\mu b_\nu)\dot{x}^\mu \dot{x}^\nu > 0\} \cap \mathcal{T}_x^A. \tag{15}$$

Properties:
1. $\mathcal{T}_x^L$ - "sharper" cone than $\mathcal{T}_x^A$.
2. on $\mathcal{T}_x^L$: $B < 0$.



Variant: Modified Randers metrics (Heefer&Fuster 2023):

$$L = \epsilon(\sqrt{|A|} - |B|)^2 = A\left(1 - \sqrt{s}\right)^2 \tag{16}$$

$\longrightarrow$ allows for

$$\langle b, b \rangle \in (-1, 1) \tag{17}$$

(and has well defined *past timelike cones*).

2. Bogoslovsky-Kropina (VSR/VGR, m-Kropina) metrics

$$L = A^{1-q} B^{2q} = A s^{q}. \tag{18}$$

Proposition 2: L - Finsler spacetime structure in precisely one of the following cases:

(i) $\langle b, b \rangle > 0$, $q \in [-1, 1)$. In this case:

$$\mathcal{T}_x^L = \mathcal{T}_x^A. \tag{19}$$

(ii) $\langle b, b \rangle = 0$, $q \in (-1, 1)$. In this case:

$$\mathcal{T}_x^L = \mathcal{T}_x^A. \tag{20}$$

(iii) $\langle b, b \rangle < 0$, $q \in (0, 1)$. In this case:

$$\mathcal{T}_x^L = \mathcal{T}_x^A \cap \{B > 0\}. \tag{21}$$

3. Generalized m-Kropina metrics:

$$L(x, \dot{x}) = As^{-p}\,(k + ms)^{p+1}\,, \tag{22}$$

where $p, k, m \in \mathbb{R}$ - const.

Proposition 3: L - Finsler spacetime structure if and only if

$$k > 0, \quad m < 0 \tag{23}$$

and one of the following occurs:
(i) $\langle b, b\rangle \geq 0,\ p \in (-1, 1]$. In this case:

$$\mathcal{T}_x^L = \mathcal{T}_x^A \cap \left\{kA + mB^2 > 0\right\}. \tag{24}$$

(ii) $\langle b, b\rangle < 0,\ p \in (-1, 0)$. In this case:

$$\mathcal{T}_x^L = \mathcal{T}_x^A \cap \left\{kA + mB^2 > 0\right\} \cap \{B > 0\}\,. \tag{25}$$

4. Exponential metrics:

$$L = Ae^{P(s)}. \tag{26}$$

Proposition 4: L - spacetime structure iff :

$$\lim_{s\to\infty} \frac{e^{P(s)}}{s} = 0, \tag{27}$$

$$1 - sP' > 0, \qquad 1 - sP' > (s - \langle b,b\rangle)\left(s\left(P'\right)^2 + 2sP'' + P'\right), \quad \forall s \in I.$$

✓ **Concrete example:**

$$\begin{aligned} A &= \eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}, B = \langle b,b\rangle\, \dot{x}^0 \\ \Psi &: [\langle b,b\rangle, \infty) \to \mathbb{R}_+, \qquad \Psi(s) = e^{k - \frac{\langle b,b\rangle^2}{2s^2}}. \end{aligned}$$

6 Isometries of (α, β)-metrics

Isometry (symmetry):

$$\varphi : M \to M, \qquad d\varphi(L) = 0. \tag{28}$$

Generators: $\xi \in \mathcal{X}(M)$ - **Killing vector field (symmetry)** for L :

$$\xi^C(L) = 0, \tag{29}$$

where $\xi^C = \xi^\mu \partial_\mu + \xi^\mu_{,\nu} \dot{x}^\nu \dot{\partial}_\mu$ - complete lift of ξ to TM.

For (α, β)-metrics

$$L = A\Psi(s), s = \frac{B^2}{A}, \tag{30}$$

$\Rightarrow$ the Killing vector condition:

$$\xi^C(A)\left(\Psi - s\Psi'\right) + 2\Psi' B \xi^C(B) = 0, \tag{31}$$

where

$$\xi^C(A) = \left(\mathfrak{L}_\xi a_{ij}\right)\dot{x}^i\dot{x}^j, \quad \xi^C(B) = \left(\mathfrak{L}_\xi b_i\right)\dot{x}^i.$$

Classes of symmetries:

1. **Trivial symmetries**: ξ - Killing vector field for A:

$$\xi^C(A) = 0 \quad (\Rightarrow \xi^C(B) = 0). \tag{32}$$

2. **Nontrivial symmetries:** $\xi^C(A) \neq 0$.

Theorem 2: $L = A\Psi(s)$ admits nontrivial symmetries $\xi \Leftrightarrow \exists\ \kappa$, m_1, m_2, n_1, n_2 - smooth functions of x only (with κ not identically zero) such that:

$$\Psi(s) = \begin{cases} cs^{\frac{m_1}{m_1 - 2n_2}} \left| m_2 s + m_1 - 2n_2 \right|^{\frac{-2n_2}{m_1 - 2n_2}}, & if \quad m_1 \neq 2n_2 \\ cse^{\frac{-2n_2}{m_2 s}}, & if \quad m_1 = 2n_2, \quad m_2 \neq 0 \end{cases} \tag{33}$$

and:

$$\xi^C(B) = \kappa n_2 B, \qquad \xi^C(A) = \kappa\left(m_1 A + m_2 B^2\right). \tag{34}$$

✓ **Concrete example: (curved) Bogoslovsky-Kropina metric** on $\mathbb{R}^4$**:**

$$A = e^{2qx^0}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \quad B = e^{(q-1)x^0}\left(\dot{x}^0 + \dot{x}^1\right),$$

$$L = A\left(\frac{B^2}{A}\right)^q, \ \ q \in (0,1)\,.$$

That is: $n_2 = q-1, \quad m_1 = 2q, \quad m_2 = 0, \quad \kappa = c = 1$. Then

$$\xi := (1,0,0,0)\,. \tag{35}$$

obeys:

$$\xi^C(A) = 2qA \neq 0, \quad \xi^C(B) = (q-1)\,B$$

$\Rightarrow$ ξ is a nontrivial Killing vector field for L.

7 Conclusion

- For (α, β)-metric deformations L of Lorentzian metrics, we have determined the conditions such that they admit a *smooth distribution of convex cones* $\mathcal{T}_x^L$, with null boundary.

- We proved that there exist (curved) (α, β)-metric spacetimes admitting symmetries that are not necessarily symmetries of the underlying Lorentzian metric.

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Thank you !