Doubly special relativity as a road to quantum gravity

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Motivation

Space-time structure

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- If fundamental constituents of matter exist, does the same happen for spacetime?
- Do space "atoms" exist?

QFT and GR: incompatibilities

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- In QFT, one assumes a given spacetime and studies in detail the properties and motion of particles in it
- In GR, one assumes that the properties of matter and radiation are given and describes in detail the resultant spacetime (curvature)
- A QGT should be valid at any energy, but an interaction mediated by spin-2 particle (same equations of GR) is not renormalizable

Why do we need a QGT?

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- Study of the first moments of the universe
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- Answers$\rightarrow$ QGT

- Attempts of unification: string theory, loop quantum gravity, supergravity, causal set theory...

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- In most of them a minimal length appears $\Longrightarrow$ Planck length (l_P)??
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- Problem: there are no experimental evidences of a fundamental QGT

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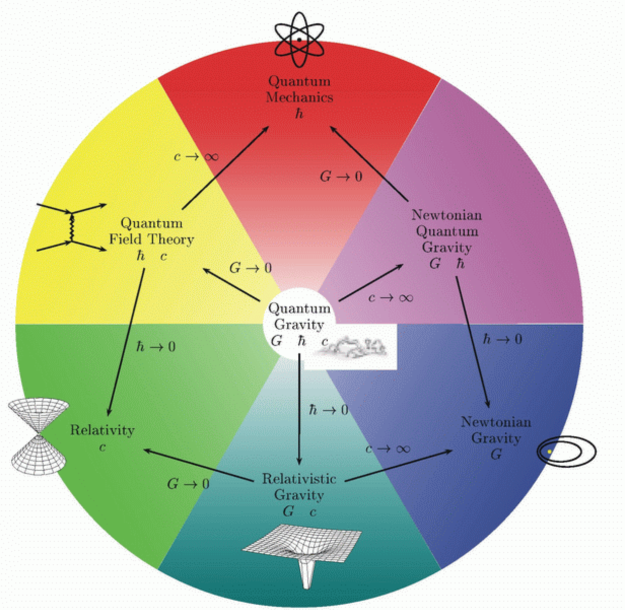
- Symmetries? $\rightarrow$ LI should be broken/deformed at Planckian scales
- New effects $\rightarrow$ Micro black holes creation?
- Spacetime can be regarded as a "foam"

Spacetime: the last frontier

Towards a quantum gravity theory



Towards a quantum gravity theory

- We can obtain l_P, t_P and M_P

$$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1{,}6 \times 10^{-35}\,\mathrm{m}$$

$$t_P = \sqrt{\frac{\hbar G}{c^5}} = 5{,}4 \times 10^{-44}\,\mathrm{s}$$

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- No quantum or gravitational effects but

$$M_P = \lim_{\hbar, G \to 0} \sqrt{\frac{\hbar c}{G}} \neq 0$$

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- There is a privileged observer $\rightarrow$ physical laws depending on the observer
- Formulated in the quantum field theory framework $\rightarrow$ standard model extension (SME)

Doubly Special Relativity (DSR)

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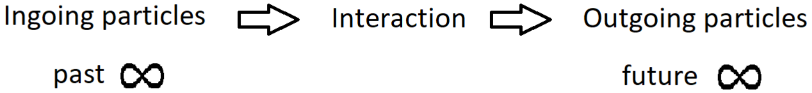
- There is a relativity principle
- Two invariants in every inertial frame: speed of light c and Planck length l_P

Kinematics

Ingoing particles $\Rightarrow$ Interaction $\Rightarrow$ Outgoing particles

past ∞ future ∞

Kinematics



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Ingoing particles $\Rightarrow$ Interaction $\Rightarrow$ Outgoing particles

past ∞ future ∞

- Ingoing and outgoing particles movement is described by the dispersion relation
- In the interaction, the conservation of total momentum holds

- Dispersion relation

$$C(k) = k_0^2 - \vec{k}^2 = m^2$$

Kinematics in SR

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- If there is a relativity principle, a different observer would also see

$$k'_\mu = p'_\mu + q'_\mu$$

i.e., the conservation of momenta viewed from a different reference frame

- Dispersion relation

$$C(k) = k_0^2 - \vec{k}^2 + \frac{k_0^3}{\Lambda} + ... = m^2$$

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- Therefore, there is a privileged reference frame (isotropic CMB radiation)

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- Dispersion relation and conservation law compatible with relativity principle $\rightarrow$ deformed Lorentz transformations

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- Noncommutative spacetime

$$\left[\tilde{x}^i, \tilde{x}^0\right] \,=\, i\frac{\tilde{x}^i}{\Lambda}$$

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but there is a deformed conservation law

$$(p \oplus q)_\mu^{\text{Snyder}} = p_\mu \left(\sqrt{1 + \frac{q^2}{\Lambda^2}} + \frac{p_\mu \eta^{\mu\nu} q_\nu}{\Lambda^2 \left(1 + \sqrt{1 + p^2/\Lambda^2}\right)} \right) + q_\mu$$

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- Phenomenology? $\rightarrow$ Really precise experiments or amplifications at low energies

LIV vs DSR phenomenology

Properties	LIV	DSR
Relativity Principle	✗	✓
Threshold energy modification	✓	✗
Processes not allowed in SR	✓	✗
Time delay of massless particles	✓	✗

LIV vs DSR phenomenology

Theory	Threshold in SR	Correction term	Energy
LIV	$\frac{m^2}{E_u^2}$	$\frac{E_u}{\Lambda}$	$E_u^3 \sim \Lambda m^2$
DSR	$\frac{m^2}{E_u^2}$	$\frac{m^2}{E_u \Lambda}$	$E_u \sim \Lambda$

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DSR only observable if $\Lambda \ll M_p$!

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- For a n dimensional space, there can be at most $n(n+1)/2$ isometries
- Maximally symmetric spaces: Minkowski (flat), de Sitter (hyperboloid), anti de Sitter (sphere)
- In Minkowski spacetime, translations and Lorentz transformations are isometries

$$\begin{aligned} D(x_2, x_2 + x_1) &= D(0, x_1) \\ D(0, x_1') &= D(0, x_1) \end{aligned}$$

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- Problem $\rightarrow$ not clear how to implement the relativity principle

Our perspective [Carmona et al. (2019)]

- Dispersion relation $\rightarrow$ Squared distance from the origin to k [Amelino-Camelia et al. (2011)]

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- Only a maximally symmetric momentum space (MSS) satisfies this! $\rightarrow$ Minkowski, de Sitter or anti de Sitter

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- Lorentz transformations $\rightarrow$ 6 isometries of the metric forming a subgroup

Construction of kinematics: de Sitter momentum space

- Start by a momentum metric

$$g_{00}(p) = 1\,, \quad g_{0i}(p) = g_{i0}(p) = \frac{p_i}{2\Lambda}\,, \quad g_{ij}(p) = -\delta^i_j e^{-p_0/\Lambda} + \frac{p_i p_j}{4\Lambda^2}$$

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- Compute the Casimir using [Relancio and Liberati (2020)]

$$C_{\rm D}(p) = f^\mu g_{\mu\nu}(p) f^\nu\,, \qquad f^\mu(p) := \frac{1}{2}\frac{\partial C_{\rm D}(p)}{\partial p_\mu}$$

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$$g_{\mu\nu}(p \oplus q) = \frac{\partial (p \oplus q)_\mu}{\partial q_\rho} g_{\rho\sigma}(q) \frac{\partial (p \oplus q)_\nu}{\partial q_\sigma}$$

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- Using this metric one obtains the same kinematics of κ-Poincaré in the symmetric basis!

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- κ-Poincaré in classical basis [Borowiec and Pachol (2010)]

$$(p \oplus q)_\mu^{\kappa-\textbf{Poincaré}} = p_\mu \left(\sqrt{1 + \frac{q^2}{\Lambda^2}} + \frac{q_0}{\Lambda} \right) + q_\mu + n_\mu \left[\frac{\sqrt{1 + p^2/\Lambda^2} - p_0/\Lambda}{1 - \vec{p}^2/\Lambda^2} \left(q_0 + \frac{q_\alpha \eta^{\alpha\beta} p_\beta}{\Lambda} \right) - q_0 \right]$$

where $n_\mu := (1, 0, 0, 0)$.

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Introduction: Klein–Gordon and Dirac equations

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- This is the analogue version of Hamiltonian in classical mechanics

$$H = \frac{\vec{p}^2}{2m} + V$$

- The generalization to a relativistic setting is the Klein–Gordon equation

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- A solution to this equation is a plane wave. Making a Fourier transformation

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  we find

$$(p^2-m^2)\tilde{\phi}(p)\,=\,0$$

Introduction: Klein–Gordon and Dirac equations

- The generalization to a relativistic setting is the Klein–Gordon equation

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- This is the onshell condition, the relativistic Hamiltonian

- The Dirac equation describes a spin 1/2 particle

$$\left(i\gamma^\mu\partial_\mu - m\mathbb{1}_4\right)\psi(x) \,=\, 0$$

where γ^μ are the Dirac matrices

$$\gamma^0 \,=\, \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}\,, \qquad \gamma^i \,=\, \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

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- By "squaring" this equation we find the Klein–Gordon equation

$$(\gamma^\mu p_\mu - m\mathbb{1}_4)\,(\gamma^\mu p_\mu + m\mathbb{1}_4) \,=\, (p^2 - m^2)\mathbb{1}_4$$

- In curved spacetimes, Klein–Gordon equation is

$$\left(C(x,p)-m^2\right)\phi(p) \,=\, \left(g^{\mu\nu}(x)p_\mu p_\nu - m^2\right)\phi(p) \,=\, 0$$

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Deformed Klein–Gordon equation: construction

- Klein–Gordon and Dirac equations already obtained in Hopf algebras [Lukierski et al. (1992); Nowicki et al. (1993)]

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- Our aim $\rightarrow$ geometrical derivation of these equations [Franchino-Viñas and Relancio (2022)]
- We are able to reproduce them from a curved momentum space!

Deformed Klein–Gordon equation: construction

- Klein-Gordon equation derived from the Casimir (squared distance)

$$\left(\Lambda^2 \operatorname{arccosh}^2\left(\cosh\left(\frac{p_0}{\Lambda}\right) - \frac{\vec{p}^2}{2\Lambda^2}\right) - m^2\right)\phi(p) = 0$$

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- Different Casimirs: possible different behavior at ultraviolet regime.

Deformed Klein–Gordon equation: invariance

- Action in momentum space

$$S_{\mathrm{KG}} := \int \mathrm{d}^4 p \sqrt{-g}\, \phi^*(p) \left(C_{\mathrm{D}}(p) - m^2 \right) \phi(p)$$

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- $\sqrt{-g}$ guarantees invariance under a change of momentum basis.
- Invariance under deformed Lorentz transformations of the metric assuming the field transforms as a scalar

$$\phi'(p') = \phi(p)$$

since

$$C_{\mathrm{D}}(p) = C_{\mathrm{D}}(p')$$

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- The new gamma matrices satisfy

$$\{\underline{\gamma}^{\mu}, \underline{\gamma}^{\nu}\} = 2 g^{\mu\nu}(p) \mathbb{1}$$

Deformed Dirac equation: formulation

- This equation can be obtained from the action

$$S_{\mathrm{Dirac}} := \int \mathrm{d}^4 p \sqrt{-g}\,\bar{\psi}(-p)\left(\underline{\gamma}^{\mu} f_{\mu}(p) - m\right)\psi(p)$$

Deformed Dirac equation: formulation

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- Klein–Gordon equation is obtained straightforwardly from

$$\left(\underline{\gamma}^{\nu} f_{\nu}(p) - m\right) \left(\underline{\gamma}^{\nu} f_{\nu}(p) + m\right) = C_{\mathrm{D}}(p) - m^2$$

- Invariant under deformed Lorentz transformations

Deformed Dirac equation: invariance and symmetries

- Invariant under deformed Lorentz transformations
- Invariant under change of momentum coordinates

Deformed Dirac equation: invariance and symmetries

- Invariant under deformed Lorentz transformations
- Invariant under change of momentum coordinates
- Discrete symmetries

$$\begin{aligned} \mathcal{P}_0 &: = \mathrm{i}\gamma^0\,, & \tilde{\psi}_{\mathcal{P}} &: = \mathrm{i}\gamma^0\tilde{\psi}(p_0, -\vec{p})\,, \\ \mathcal{T}_0 &: = \mathrm{i}\gamma^1\gamma^3\mathcal{K}\,, & \tilde{\psi}_{\mathcal{T}} &: = \mathrm{i}\gamma^1\gamma^3\tilde{\psi}^*(p_0, -\vec{p})\,, \\ \mathcal{C}_0 &: = \mathrm{i}\gamma^2\mathcal{K}\,, & \tilde{\psi}_{\mathcal{C}} &: = \mathrm{i}\gamma^2\tilde{\psi}^*(-p)\,. \end{aligned}$$

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- Invariant under $\mathcal{P}$ and $\mathcal{T}$
- Invariant under $\mathcal{C}$ when $\Lambda \to -\Lambda$

Deformed Dirac equation: choice of tetrad

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- The composition law identifies one and only one tetrad:

$$g_{\mu\nu}\left(p \oplus q\right) = \frac{\partial \left(p \oplus q\right)_\mu}{\partial q_\rho} g_{\rho\sigma}\left(q\right) \frac{\partial \left(p \oplus q\right)_\nu}{\partial q_\sigma}$$

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so for $q \rightarrow 0$

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$$e_\mu{}^a(p) := \delta^a_\nu \left.\frac{\partial\left(p\oplus q\right)_\mu}{\partial q_\nu}\right|_{q\rightarrow 0}$$

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- We can construct the Dirac equation for different relativistic kinematics!

Deformed Dirac equation in κ-Poincaré

- For the symmetric basis we find

$$\mathcal{D}_{\mathrm{D}}^{(S)} := \frac{\sqrt{\frac{C_{\mathbf{D}}^{(S)}(p)}{\Lambda^2}}}{2\Lambda \sinh\left(\sqrt{\frac{C_{\mathbf{D}}^{(S)}(p)}{\Lambda^2}}\right)} \left[2\Lambda e^{-\frac{p_0}{2\Lambda}}\gamma^i p_i + \gamma^0\left(2\Lambda^2 \sinh\left(\frac{p_0}{\Lambda}\right) - \vec{p}^2\right)\right]$$

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- If we use instead $C_{\mathrm{A}}^{(S)}(p)$

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which is the same result obtained in Hopf algebras! [Nowicki et al. (1993)]

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- Our construction leads to

$$\left(\mathcal{D}^{(S)}_{\mathrm{D}}\right)^2 = C^{(S)}_{\mathrm{D}}$$

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Conclusions: curved momentum space

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- We obtain the κ-Poincaré kinematics from a de Sitter momentum space
- Other possible kinematics can be obtained though this framework and also for anti de Sitter

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- Future work: include interactions

Thanks for your attention!!!

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