The Schrödinger equation is incomplete: Hamiltonian pictures of QFT in curved space-times

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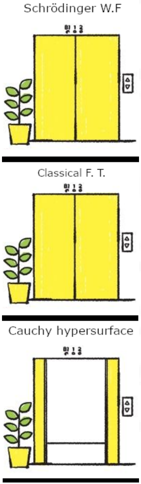
Presentation Outline

Ground floor: space+time

In globally hyperbolic spacetimes the spacetime manifold is diffeomorphic to $\Sigma \times \mathbb{R}$



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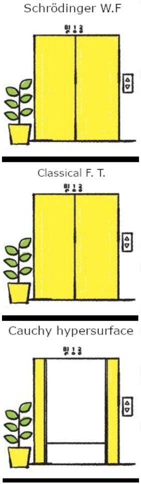
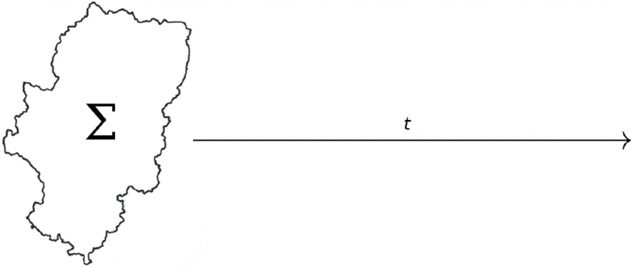
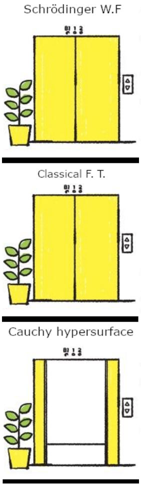


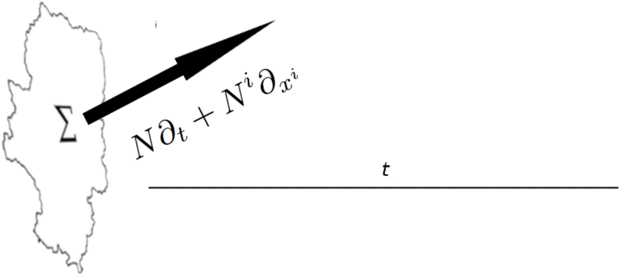


For simplicity Σ is compact.

Ground floor: space+time



We can develop ADM formalism of general relativity only on Σ_t, the phase space of the theory is $T^*Riemm(\Sigma)$ with coordinates $(h^{ij}(x), \pi_{ij}(x))$ and the diffeomorphism is fully implemented by Lapse and Shift functions N, N^i.



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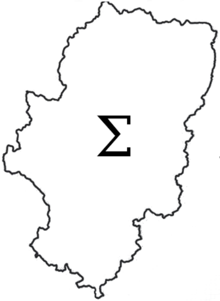
First floor: Classical Field theory



Klein-Gordon Theory on Minkowsky.

$$\frac{\partial^2}{\partial t^2}\varphi(t,\vec{x}) = (\triangle - m^2)\varphi(t,\vec{x})$$

$$\varphi(0,\vec{x}) \in C_c^\infty(\Sigma),$$

$$\frac{d}{dt}\varphi(0,\vec{x}) \in C_c^\infty(\Sigma).$$

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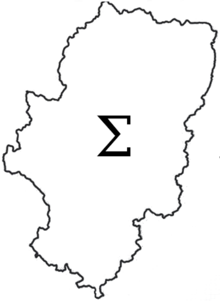

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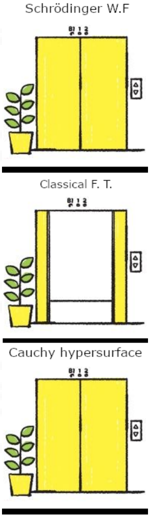
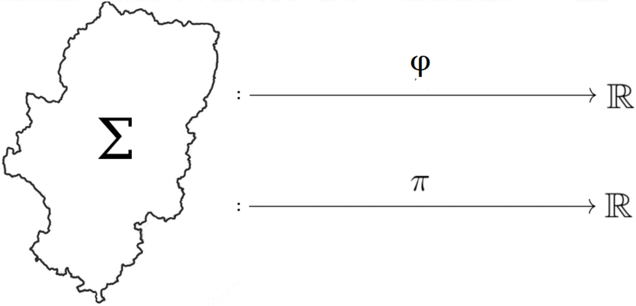


First floor: Classical Field theory

The only relevant information is fixed over Σ is the field configuration $\varphi(\vec{x})$, and its time derivative $\pi(\vec{x})$.



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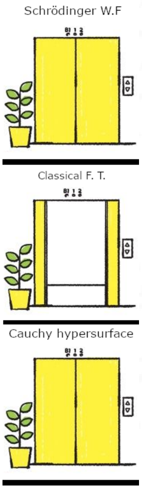


The dynamics of the coupled system is given by a Hamiltoninan

$$H = \int d^3x \left(N\mathscr{H}^{(T)} + N^i \mathscr{H}^{(T)} \right)$$

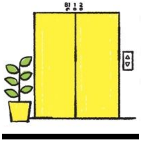
$$\mathscr{H}^{(T)} = \mathscr{H} + \mathscr{H}^{(M)}$$

$$\mathscr{H}_i^{(T)} = \mathscr{H}_i + \mathscr{H}_i^{(M)}$$

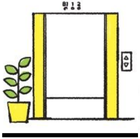
$$\{,\} = \{,\}_M + \{,\}_G$$

First floor: Classical Field theory

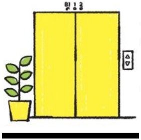
Schrödinger W.F



Classical F. T.



Cauchy hypersurface



$$\mathscr{H}^{(T)} = \mathscr{H} + \mathscr{H}^{(M)}$$

$$\mathscr{H}_i^{(T)} = \mathscr{H}_i + \mathscr{H}_i^{(M)}$$

For the gravitational part this is

$$\mathscr{H} = \frac{1}{2}\frac{(2\kappa)}{\sqrt{h}} G_{ijkl}\pi^{ij}\pi^{kl} - (2\kappa)^{-1}\sqrt{h}R(h)$$

$$\mathscr{H}_{iG} = -2D_j\pi_i^j = -2D_j(h_{i\alpha}\pi^{\alpha j}) = -2h_{i\alpha}D_j\pi^{\alpha j}$$

While for the matter part we get

$$\mathscr{H}^{(M)} = \sqrt{h}\frac{1}{2}[\pi^2 + h^{ij}D_i\varphi D_j\varphi + m^2\varphi^2]$$

$$\mathscr{H}_i^{(M)} = \sqrt{h}\pi D_i\varphi$$

First floor: Classical Field theory

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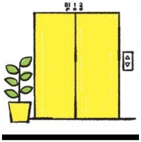

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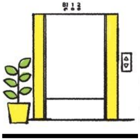

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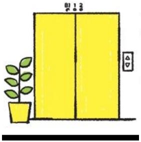

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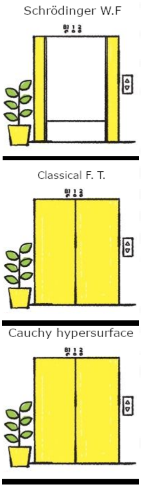
$$\{,\} = \{,\}_M + \{,\}_G$$

$$\{,\}_G = \int_\Sigma dvol_\Sigma \ \left(\delta_{h_{ij}(x)} \otimes \delta_{\pi^{ij}(x)} - \delta_{\pi^{ij}(x)} \otimes \delta_{h_{ij}(x)} \right),$$

$$\{,\}_M = \int_\Sigma d^3x \ \left(\delta_{\varphi(x)} \otimes \delta_{\pi(x)} - \delta_{\pi(x)} \otimes \delta_{\varphi(x)} \right).$$

Second floor: Quantum field theory



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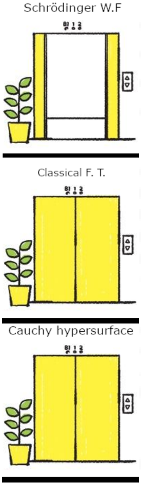
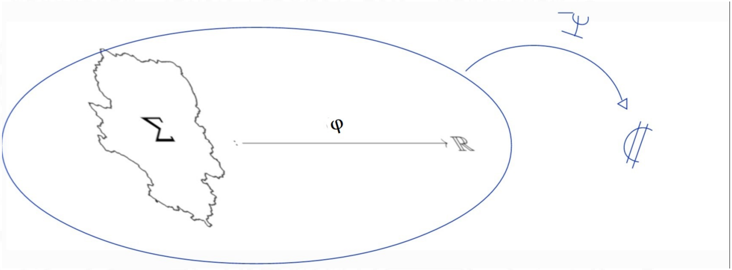


To describe a quantum pure state Ψ we must construct a Hilbert space of $\Psi \in L^2(D\mu)$ functions.

- The field configurations over Σ i.e. $\varphi \in C_c^\infty(\Sigma)$ must be in the domain of Ψ.



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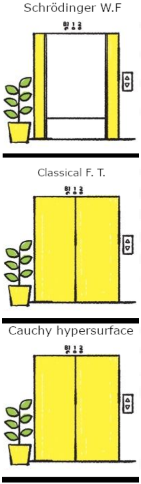

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- The measure $D\mu$ is gaussian probability and is part of the quantum vacuum.

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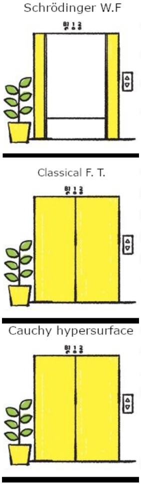


To describe a quantum pure state Ψ we must construct a Hilbert space of $\Psi \in L^2(D\mu)$ functions.

- The field configurations over Σ i.e. $\varphi \in C_c^\infty(\Sigma)$ must be in the domain of Ψ.
- The measure $D\mu$ is gaussian probability and is part of the quantum vacuum.

No such a Borel Probability measure exist over $C_c^\infty(\Sigma)$ but Bochner-Minlos theorem ensures its existence over its strong dual $D'(\Sigma)$.

$$\int_{D'(\Sigma)} e^{i\langle\varphi,\xi\rangle} D\mu(\varphi) = e^{-\frac{1}{2}\Delta(\xi,\xi)}$$

Where $\Delta : C_c^\infty(\Sigma) \times C_c^\infty(\Sigma) \to \mathbb{R}$ is a continuous bilinear.

Hybrid Geometrodynamics

Hybrid Geometrodynamics:
A Hamiltonian description of classical gravity coupled to quantum matter.

J. L. Alonso,[1,2,3] C. Bouthelier-Madre,[1,2,3] J. Clemente-Gallardo,[1,2,3] and D. Martínez-Crespo[1,3]

arXiv:2307.00922v1 [gr-qc] 3 Jul 2023

Direct quantization

Lets attempt the naive quantization procedure

$$\mathcal{Q}[\varphi(x)] = \varphi(x) \text{ and } \mathcal{Q}[\pi(x)] = -i\frac{\delta}{\delta\varphi(x)}$$

$$i\frac{d}{dt}\Psi(\varphi) = \mathcal{Q}[H]\Psi(\varphi)$$

Direct quantization

$$i\frac{d}{dt}\Psi(\varphi) = \mathcal{Q}[H]\Psi(\varphi)$$

This evolution operator leads to norm loss in the evolution for a variety of non trivial space-times

Classical versus quantum completeness

Stefan Hofmann[1,*] and Marc Schneider[1,†]

[1] *Arnold Sommerfeld Center for Theoretical Physics, Theresienstraße 37, 80333 München*

(Dated: June 26, 2015)

The notion of quantum-mechanical completeness is adapted to situations where the only adequate description is in terms of quantum field theory in curved space-times. It is then shown that Schwarzschild black holes, although geodesically incomplete, are quantum complete.

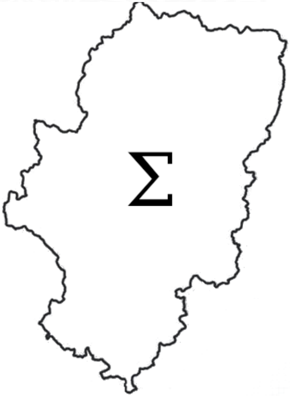
PACS numbers: 03.65.-w, 03.65.Db, 03.70.+k, 04.20.Dw, 11.10.-z, 11.10.Ef.

Quantization on a Cauchy hypersurface

We will analyze the problem of how to implement a dynamical equation and quantization procedure that preserves the norm in the evolution.

Quantization on a Cauchy hypersurface

First we pause time and describe the quantization procedure over Σ



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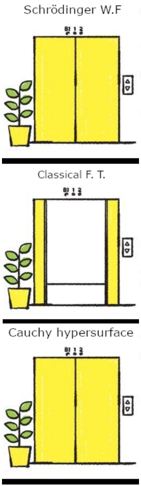

$$\omega_M = \int_\Sigma d^3x \; d\pi(x) \wedge d\varphi(x)$$

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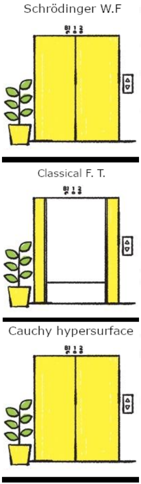

$$\omega_M = \int_\Sigma d^3x \; d\pi(x) \wedge d\varphi(x)$$

For quantization we need a Hilbert space, then we need a Kähler structure (μ, ω, J) such that $\mu(,) = \omega(, -J)$

$$-J_{\mathcal{M}_F} = (\partial_{\varphi^y}, \partial_{\pi^y}) \begin{pmatrix} A_x^y & \Delta_x^y \\ D_x^y & -(A^t)_x^y \end{pmatrix} \begin{pmatrix} d\varphi^x \\ d\pi^x \end{pmatrix}$$

with $\Delta_{\mathbf{xy}} > 0 > D_{\mathbf{xy}}$ and $J^2_{\mathcal{M}_F} = -\mathbb{1}$

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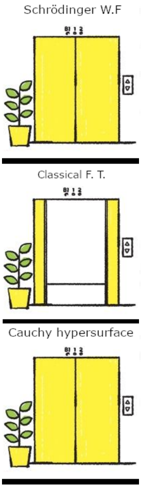

$$\omega_M = \int_\Sigma d^3x \; d\pi(x) \wedge d\varphi(x)$$

In Klein Gordon

$$\frac{d}{dt}\begin{pmatrix} \varphi_x \\ \pi_y \end{pmatrix} = \left\{ \begin{pmatrix} \varphi_x \\ \pi_y \end{pmatrix}, H \right\}_{\mathcal{M}_F} = F \begin{pmatrix} \varphi_x \\ \pi_y \end{pmatrix}$$

$$F = \begin{pmatrix} N^i D_i & N \\ -N D^i D_i - (D^i N) D_i + N m^2 & N^i D_i + D_i N^i \end{pmatrix}$$

$$J_{\mathcal{M}_F} = |F|^{-1} F,$$

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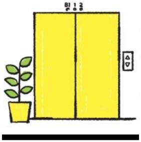

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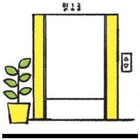

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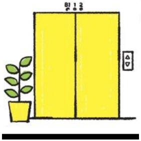

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$$\mu_{\mathcal{M}_F} = (d\varphi^y, d\pi^y) \begin{pmatrix} \Delta_{yx} & -A_{yx} \\ -A^t_{yx} & -D_{yx} \end{pmatrix} \begin{pmatrix} d\varphi^x \\ d\pi^x \end{pmatrix}$$

Schrödinger representation for a scalar field on curved spacetime

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and Perimeter Institute for Theoretical Physics, 35 King Road North, Waterloo, Ontario, Canada N2J 2W9

Jerónimo Cortez† and Hernando Quevedo‡
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, México D.F. 04510, México
(Received 25 July 2002; published 30 October 2002)

Geometric Quantization

The geometric quantization procedure provides a recipe, up to ordering problems, for the phase space of the theory

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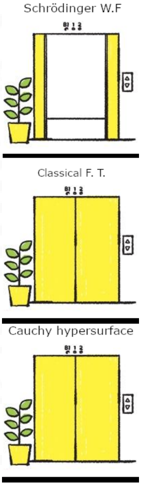
$$\int_{\mathcal{N}'} D\mu_S(\varphi^{\mathbf{x}}) e^{i\xi_x \varphi^x} = e^{-\frac{1}{4}\xi_x \Delta^{xy} \xi_y}.$$



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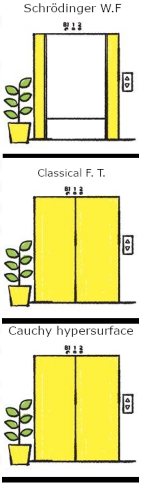
$$L^2(D'(\Sigma), D\mu_S), \text{ with } \int_{D'(\Sigma)} D\mu_S(\varphi^{\mathbf{x}}) e^{i\xi_x \varphi^x} = e^{-\frac{1}{4}\xi_x \Delta^{xy} \xi_y}.$$

Fot the operators, with K the inverse of Δ, we get

$$\mathcal{Q}(\varphi^x)\Phi(\varphi^{\mathbf{x}}) = \varphi^x \Phi(\varphi^{\mathbf{x}}),$$
$$\mathcal{Q}(\pi_y)\Phi(\varphi^{\mathbf{x}}) = \left(-i\partial_{\varphi^y} + i\varphi^z K_{zy} - \varphi^x (KA)_{xy}\right)\Phi(\varphi^{\mathbf{x}}).$$



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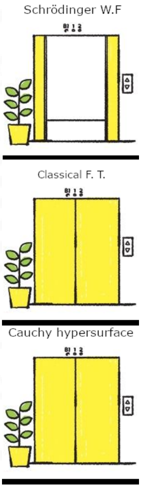
Geometric flavours of Quantum Field theory on a Cauchy hypersurface. Part I: Geometric quantization

José Luis Alonso[1,2,3], Carlos Bouthelier-Madre[1,2,3], Jesús Clemente-Gallardo[1,2,3], and David Martínez-Crespo[1,3]

arXiv:2306.14844v1 [math-ph] 26 Jun 2023



Time dependence

The norm loss comes from the fact that both $L^2(\mathcal{N}', D\mu_S)$ and $\mathcal{Q}$ depend on time

$$J_{\mathcal{M}_F} = (\partial_{\varphi^y}, \partial_{\pi^y}) \begin{pmatrix} A_x^y & \Delta_x^y \\ D_x^y & -(A^t)_x^y \end{pmatrix} \begin{pmatrix} d\varphi^x \\ d\pi^x \end{pmatrix}$$

$$\int_{\mathcal{N}'} D\mu_S(\varphi^{\mathbf{x}}) e^{i\xi_x \varphi^x} = e^{-\frac{1}{4}\xi_x \Delta^{xy} \xi_y}.$$

$$\mathcal{Q}(\varphi^x)\Phi(\varphi^{\mathbf{x}}) = \varphi^x \Phi(\varphi^{\mathbf{x}}),$$

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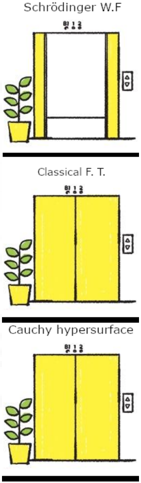
Second quantized Kähler structure

The norm loss is characterized by a second quantized Kähler structure.

$$\langle \Psi_1, \Psi_2 \rangle = \frac{\mathcal{G}(\Psi_1, \Psi_2) - i\Omega(\Psi_1, \Psi_2)}{2}.$$

$$\mathcal{J}_{\mathscr{P}} = i(d\Psi^{\phi} \otimes \partial_{\Psi^{\phi}} - d\overline{\Psi}^{\bar{\sigma}} \otimes \partial_{\overline{\Psi}^{\bar{\sigma}}})$$



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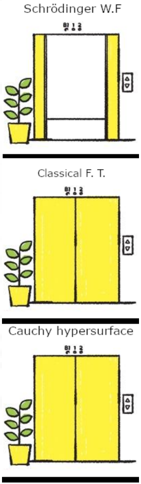
The Schrödinger equation is given by the associated Poisson bracket structure

$$\frac{d}{dt}\Psi = X_H \Psi \text{ with } X_H = \left\{\cdot, \langle \Psi, \hat{H}\Psi \rangle\right\}$$

The only allowed observables are quadratic functions $f_{\hat{G}}(\bar{\Psi}, \Psi) = \langle \Psi, \hat{G}\Psi \rangle$



Connection term

Our solution to this problem is to consider Ψ a section of a bundle $B \to \mathbb{R}$ whose locally equivalent to

$$L^2(D'(\Sigma), D\mu_s(t)) \times \mathbb{R}$$

. With this we substitute the time derivative by a covariant time derivative

$$\nabla_t \Psi = \partial_t \Psi + \Gamma \Psi$$

such that $\nabla_t \mathcal{G} = \nabla_t \Omega = \nabla_t \mathcal{J} = 0$.

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such that $\nabla_t \mathcal{G} = \nabla_t \Omega = \nabla_t \mathcal{J} = 0$.
We also need to modify the quantization procedure to get

$$\nabla_t \mathcal{Q}(F) = \mathcal{Q}(\partial_t F)$$

for a sufficient class of F assuming that the canonical coordinates $\varphi(x), \pi(x)$ are independent on time.

If we assume that the canonical coordinates $\varphi(x), \pi(x)$ are independent on time, we should treat them as exchangeable coordinates in our construction.

$$J_{\mathcal{M}_F} = (\partial_{\varphi^y}, \partial_{\pi^y}) \begin{pmatrix} A_x^y & \Delta_x^y \\ D_x^y & -(A^{t})_x^y \end{pmatrix} \begin{pmatrix} d\varphi^x \\ d\pi^x \end{pmatrix}$$

$$\int_{\mathcal{N}'} D\mu_S(\varphi^{\mathbf{x}}) e^{i\xi_x \varphi^x} = e^{-\frac{1}{4}\xi_x \Delta^{xy} \xi_y}.$$

$$\mathcal{F} \Big\downarrow$$

$$\int_{D'(\Sigma)} D\nu_M(\pi^{\mathbf{x}}) e^{i\xi_x \pi^x} = e^{\frac{1}{4}\xi_x D^{xy} \xi_y}.$$

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$$\tilde{\nabla}_t \mathcal{O} = \frac{1}{2}\left[\frac{\partial \mathcal{O}}{\partial t} + \left(\frac{\partial \mathcal{O}^\dagger}{\partial t}\right)^\dagger\right] + \mathcal{J}\frac{1}{2}\mathcal{F}^{-1}\left[\frac{\partial \hat{\mathcal{O}}}{\partial t} - \left(\frac{\partial \hat{\mathcal{O}}^\dagger}{\partial t}\right)^\dagger\right]\mathcal{F}$$

This choice preserves the Kähler structure.

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This choice preserves the Kähler structure.
To relate also the quantization we should factorize $\Psi = \Psi_0 \Psi_h$ where Ψ_h is the Hilbert space state and Ψ_0 is part of the vacuum. Asking for

$$\Psi_0 = \exp\left(-\frac{i}{2}\varphi^x (KA)_{xy} \varphi^y\right)$$

we expect to be able to recover $\nabla_t \mathcal{Q}(F) = \mathcal{Q}(\partial_t F)$

The modified schrodinger equation

$$\nabla_t \Psi = \partial_t \Psi + \Gamma \Psi = -\mathcal{J}\mathcal{Q}(H)\Psi$$

In the holomorphic picture

$$\chi_x \Delta^{xy} \xi_y = \int_\Sigma d^d x \sqrt{h}\chi(x) \sqrt{\frac{N}{-ND^i D_i - (D^i N) D_i + Nm^2}} \xi(x) \quad (1)$$

$$i\left[\frac{\partial}{\partial t} - \frac{1}{2}\phi^y K_{yz} \dot{\Delta}^{zx} \partial_{\phi^x}\right]\Psi = [\hat{H} + \frac{1}{2}\phi^y (K_{yz}\dot{\Delta}^{zx} - \frac{\dot{h}}{2h}\varphi(x)\delta^3(x-y))\partial_{\phi^x}]\Psi$$

Further reading

Geometric flavours of Quantum Field theory on a Cauchy hypersurface. Part I: Geometric quantization

José Luis Alonso[1,2,3], Carlos Bouthelier-Madre[1,2,3], Jesús Clemente-Gallardo[1,2,3], and David Martínez-Crespo[1,3]

arXiv:2306.14844v1 [math-ph] 26 Jun 2023

Geometric flavours of Quantum Field theory on a Cauchy hypersurface. Part II: Methods of quantization and evolution

José Luis Alonso[1,2,3], Carlos Bouthelier-Madre[1,2,3], Jesús Clemente-Gallardo[1,2,3]* and David Martínez-Crespo[1,3]

Further reading

Hybrid Geometrodynamics:
A Hamiltonian description of classical gravity coupled to quantum matter.

J. L. Alonso,[1,2,3] C. Bouthelier-Madre,[1,2,3] J. Clemente-Gallardo,[1,2,3] and D. Martínez-Crespo[1,3]

arXiv:2307.00922v1 [gr-qc] 3 Jul 2023

Thank You So Much!