Vacuum energy in different geometries

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Introduction

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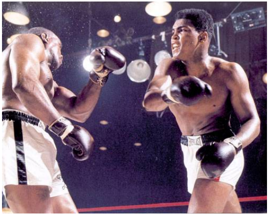

Not this!!

Introduction

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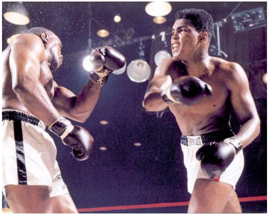


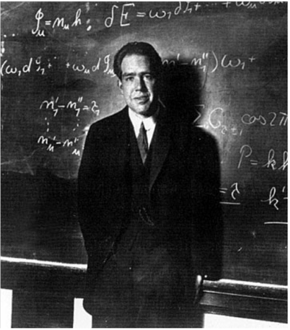
Not this!!

- It is an effect that appears between neutral objects at distances of the order of hundred micrometers down to nanometers.
- It has quantum origin with macroscopic consequences. It is due to the fluctuating fields. For that reason, it could be realized in the quantum fluctuations *of any kind of field.*
- Casimir discovered the phenomenon thinking about a situation involving fluctuating electromagnetic fields.

Hendrik Brugt Gerhard Casimir (1909-2000)

He was a dutch physicist and humanist.
During his PhD studies at the University of Leiden he would visit Copenhagen where he meet Niels Bohr and who later would contribute to bring Casimir into the right path of thinking.
Casimir was working at the Philips Research Labs when he got interested on the disagreement between theory and experiments in colloidal systems.

The trigger



Here is what happened. During a visit I paid to Copenhagen , it must have been in 1946 or 1947, Bohr asked me what I had been doing and I explained our work on van der Waals forces. "That is nice" he said, "that is something new". I then

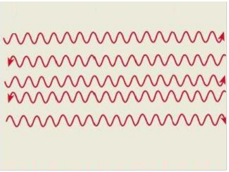
explained I should like to find a simple and elegant derivation of my results. Bohr thought this over, then mumbled something like "must have something to do with the zero-point energy". That was all, but in retrospect I have to admit that I owe much to this remark.

Casimir realized that in a similar way as we consider particles fluctuating inside atoms, he could consider electromagnetic field fluctuations in the vacuum, **the "zero point fluctuations".**

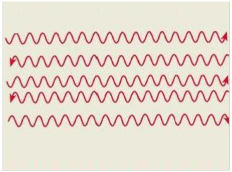
The basic nature of quantum physics is fluctuations

$$\Delta q \Delta p \geq \frac{\hbar}{2}, \qquad [q,p] = i\hbar$$

The Hamiltonian does not commute with neither of them so in an energy eigenstate both $\Delta q > 0$ and $\Delta p > 0$. Consequently $E_n = \hbar\omega(n + 1/2)$.

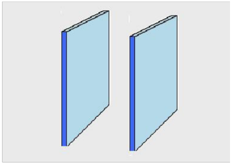


- Each point in space can be associated with a quantum fluctuation of the 'electromagnetic field'.
- The vacuum of QFT can be considered as an extremely large collection of harmonic oscillators, each with energy $E_n = \hbar\omega(n + 1/2)$.
- Ground state $n = 0$. The energy is $E_0 = \sum_k \frac{1}{2}\hbar\omega_k$, the Zero Point Energy, ZPE.

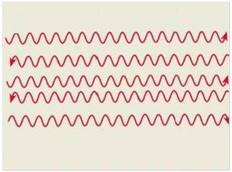


- The presence of conducting parallel plates imposes boundary conditions on the electromagnetic field

$$\vec{E}\times\hat{n}|_{plates} = 0, \quad \vec{B}\cdot\hat{n}|_{plates} = 0$$



- This modifies the frequency of the radiation field.



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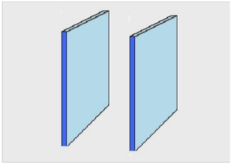
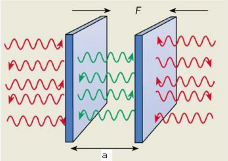


- This modifies the frequency of the radiation field.

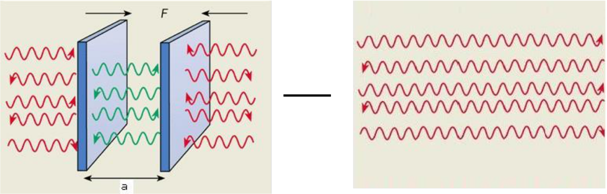
The zero point energy undergoes a change due to the presence of the parallel plates that imposes restrictions on the electromagnetic fields.



The possible modes of the EM field are now restricted by the presence of the boundaries. Consequently, the zero point energy acquire different allowed frequencies,

$$E_{0_{pp}} = \sum_k \frac{1}{2}\hbar\bar{\omega}_{k_{pp}}$$

If we now subtract the vacuum, the energy per unit area is



$$\mathcal{E}_{Cas} = \underset{s\to 0}{\text{lím}} \left[\sum_k \frac{1}{2}\hbar\bar{\omega}_{k_{pp}}(s) - \sum_k \frac{1}{2}\hbar\omega_k(s) \right] = -\frac{\pi^2}{1440a^3}$$

where s is a renormalization parameter. More of this later.

The Casimir effect gives rise to a force manifested by the change of the zero-point energy of a quantized field due to the presence of boundary conditions,

$$\mathcal{F} = -2\frac{\partial \mathcal{E}}{\partial a} = -\frac{2\pi^2}{480a^4}$$

The need to renormalize

- The fluctuations of the field $\Longrightarrow$ harmonic oscillator at each point $\omega_{\mathbf{k}} = c\sqrt{\mathbf{k}^2}$ where $\mathbf{k} = (k_x, k_y, k_z)$. Then

$$\mathcal{E}_{vac} = \sum_k \frac{1}{2}\hbar\bar{\omega}_{\mathbf{k}} = \int_{-\infty}^{\infty} \frac{1}{2}\frac{d\mathbf{k}}{(2\pi)^3}\hbar c|\mathbf{k}| = \infty$$

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- The presence of the boundaries $\Longrightarrow$ changes the frequency of vibration of the fields $\omega_{\mathbf{k}'}$. It confines the modes in the normal direction to the plane defined by the plates restricting the values of the momentum in that direction $\mathbf{k} = (k_x, k_y, \frac{\pi n}{a})$,

$$\mathcal{E}_{pp} = \sum_{k'} \frac{1}{2}\hbar\bar{\omega}_{\mathbf{k}'} = \int_{-\infty}^{\infty} \frac{1}{2}\frac{d\mathbf{k}}{(2\pi)^3}\hbar c\sqrt{\mathbf{k}_\perp^2 + \left(\frac{\pi n}{a}\right)^2} = \infty$$

and gives rise to a different infinite energy.

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• Casimir energy,

$$\mathcal{E}_{Cas} = \sum_k \frac{1}{2}\hbar\bar{\omega}_{\mathbf{k}} - \sum_{k'} \frac{1}{2}\hbar\omega_{\mathbf{k}'} = \infty - \infty$$

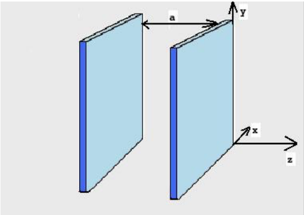
The need to renormalize

$$\mathcal{E}_{Cas} = \sum_k \frac{1}{2}\hbar\bar{\omega}_{\mathbf{k}} - \sum_{k'} \frac{1}{2}\hbar\omega_{\mathbf{k}'} = \infty - \infty$$

ALRIGHT RUTH, I ABOUT GOT THIS ONE RENORMALIZED.

Source theory

In quantum field theory we need to evaluate expressions that involve the vacuum expectation value of the fields, which relates to the Green's function by $< T\phi(x)\phi(x) >= -iG(x,x)$.



Let's start with the Lagrangian for a scalar field ϕ interacting with two delta potentials

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}V(z)\phi^2,$$

where $V(z) = \lambda\delta(z) + \lambda'\delta(z-a)$ and λ, λ' are the coupling constants.

For Dirichlet b.c. they go to infinity and $\phi(0), \phi(a) \to 0$.

The geometry of the problem allows us to write,

$$G(x,x') = \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x'}_\perp)} \int \frac{d\omega}{2\pi} e^{i\omega(t-t')} g(z,z'),$$

where the reduced green function satisfies the equation of motion,

Green's functions

$$\left[-\frac{d^2}{dz^2} + \kappa^2 + \lambda\delta(z) + \lambda'\delta(z-a)\right] g(z,z') = \delta(z-z'),$$

where

- $\kappa^2 = k_\perp^2 - \omega^2$,
- $g(z,z')$ is the reduced Green function
- there is symmetry in the transverse components.

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where

- $\kappa^2 = k_\perp^2 - \omega^2$,
- $g(z,z')$ is the reduced Green function
- there is symmetry in the transverse components.

In the absence of plates, the free Green function satisfies

$$\left(-\frac{d^2}{dz^2}+\kappa^2\right)g_0(z-z')=\delta(z-z'),$$

We can now calculate the force on the places or the energy between the place by making use of the energy momentum tensor,

$$\langle T^{\mu\nu}\rangle = \left(\partial^\mu\partial\nu' - \frac{1}{2}g^{\mu\nu}\mathcal{L}\right)\frac{1}{i}G(x,x')\bigg|_{x=x'}.$$

Two ways:

- Normal component $T_{zz'}$ Pressure on the plate at $z = a$ is the difference,

$$\mathcal{F} = \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} \int \frac{d\omega}{2\pi} \left(\langle t_{zz'} \rangle_{in} \big|_{z=z'=a} - \langle t_{zz} \rangle_{out} \big|_{z=z'=a} \right)$$

- Energy component, T_{00} Large λ corresponds to Dirichlet bc, $\mathcal{F} \sim -\frac{\pi^2}{480a^4}$

  We can also compute the energy,

$$E = \int d\mathbf{r} \langle T^{00} \rangle$$

  from where we can extract the previous shown result. We also have the relation

$$\mathcal{F} = -\frac{\partial \mathcal{E}}{\partial a}.$$

- Both expressions have to be renormalized. Regardless the renormalization method, both must give the same value.

Fractal $\longleftrightarrow$ Self-similarity

We understand by fractal a geometrical figure, in which similar patterns recur at progressively smaller and/or bigger scales.

We have worked in some self-similar configurations,

- δ-function plates positioned at points given by the series

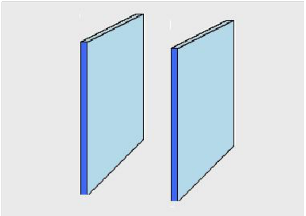
$$\sum_{n=0}^{\infty} \frac{a}{2^n} \qquad \text{and} \qquad \sum_{n=0}^{\infty} 2^n a$$

- δ-function plates located at points of a Cantor set,
- Sierpinski triangle and other geometrical figures of the same kind.

and calculate the Casimir energy in two independent ways.

Two δ-function plates

$$\mathcal{E} = \mathcal{E}_0 + \Delta\mathcal{E}_1 + \Delta\mathcal{E}_2 + \Delta\mathcal{E}_{12}(a)$$

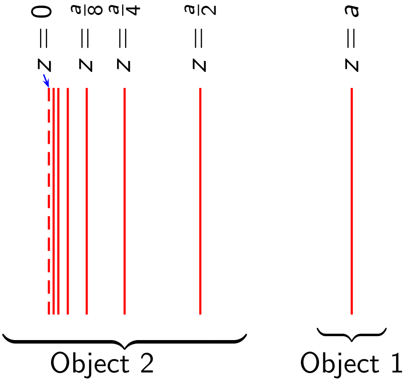


- The first three terms on the right diverge.
- If we impose Dirichlet boundary conditions, the interaction energy between the plates is

$$\Delta\mathcal{E}_{12}(a) = -\frac{\pi^2}{1440a^3}.$$

Self-similar δ-function plates at points $\frac{a}{2^n}$

Let's consider a geometric sequence of parallel plates $\frac{a}{2^n}$



$$\mathcal{E}_0 + \sum_{i=1}^{\infty} \Delta\mathcal{E}_i + \Delta\mathcal{E}(a)$$

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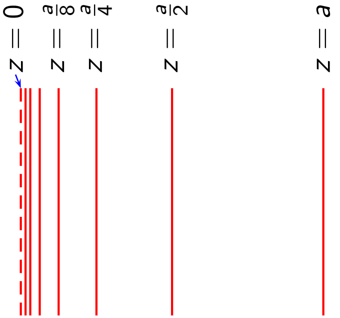

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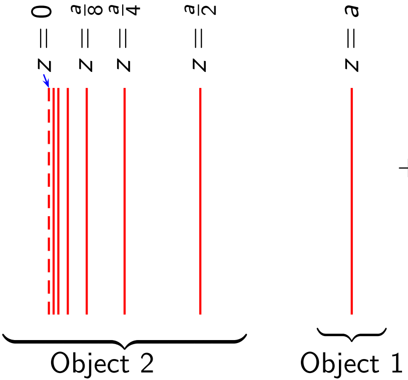

$$\mathcal{E}_0 + \sum_{i=1}^{\infty} \Delta\mathcal{E}_i + \Delta\mathcal{E}(a) = \mathcal{E}_0 + \Delta\mathcal{E}_1$$

$$+ \left(\sum_{i=2}^{\infty} \Delta\mathcal{E}_i + \Delta\mathcal{E}(a/2) \right) + \Delta\mathcal{E}_{12}(a),$$

Self-similar δ-function plates. Dirichlet b.c.

The interaction energy is a function of only a:

$$\Delta\mathcal{E}(a) = \Delta\mathcal{E}(a/2) + \Delta\mathcal{E}_{12}(a),$$

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In 3+1 D the dimension of the energy per unit area is $[L]^{-3}$ allowing us to write

$$\Delta\mathcal{E}(a/2) = 2^3\,\Delta\mathcal{E}(a).$$

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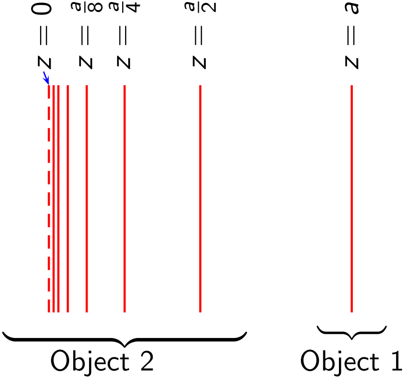
Taking this scaling into account, we get an expression similar in nature to the self-similar series on the first slide,

$$\boxed{\Delta\mathcal{E}(a) = 8\,\Delta\mathcal{E}(a) + \Delta\mathcal{E}_{12}(a)}$$

$$\Delta\mathcal{E}(a) = -\frac{1}{7}\Delta\mathcal{E}_{12}(a)$$

Self-similar δ-function plates at points $\frac{a}{2^n}$



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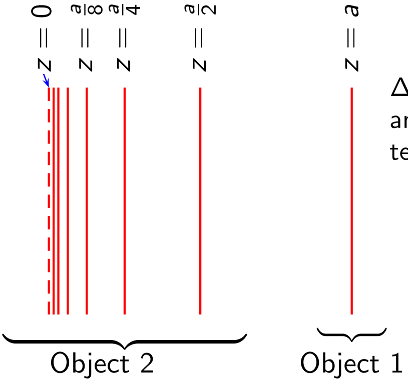

$$\Delta\mathcal{E}(a) = -\frac{1}{7}\Delta\mathcal{E}_{12}(a)$$

$\Delta\mathcal{E}_{12}(a)$ Interaction between Object 1 and Object 2, in general not easy to determine.

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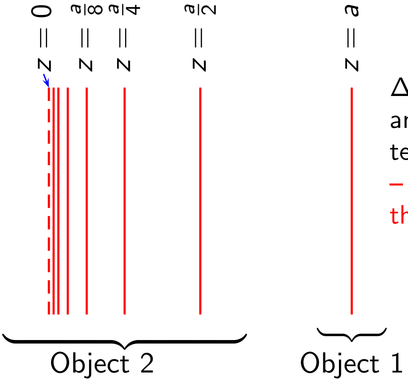

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– Dirichlet boundary conditions between the plates is assumed.

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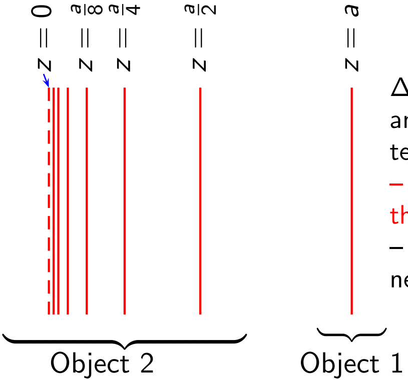


$$\Delta\mathcal{E}(a) = -\frac{1}{7}\Delta\mathcal{E}_{12}(a)$$

$\Delta\mathcal{E}_{12}(a)$ Interaction between Object 1 and Object 2, in general not easy to determine.

– Dirichlet boundary conditions between the plates is assumed.

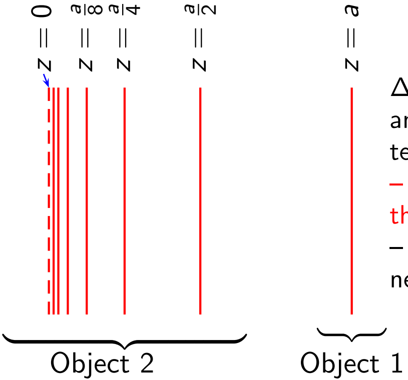
– Then, each plate interacts only with its neighbors, and the interaction energy is

$$\Delta\mathcal{E}_{ij}(L) = -\frac{\pi^2}{1440 L^3}$$

$\Delta\mathcal{E}_{12}(a)$ is the interaction energy between plates at $z = \frac{a}{2}$ and $z = a$,

$$\Delta\mathcal{E}_{12}(a) = -\frac{\pi^2}{1440(a/2)^3}$$

Self-similar δ-function plates at points $\frac{a}{2^n}$



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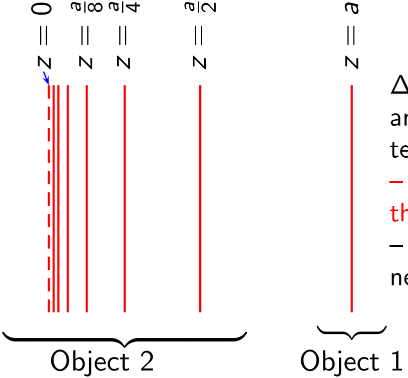

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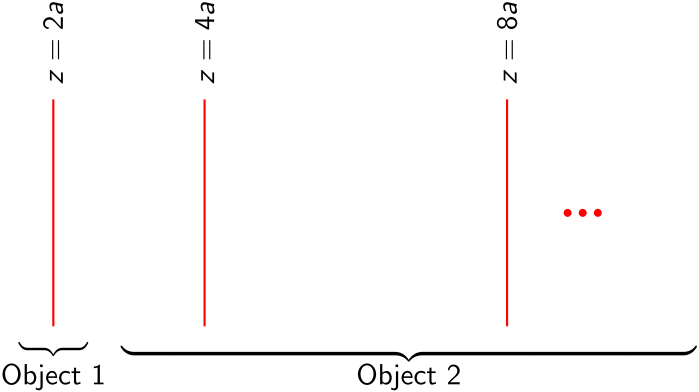
Casimir interaction energy per unit area for our set of plates

$$\boxed{\Delta\mathcal{E}(a) = +\frac{8}{7}\frac{\pi^2}{1440 a^3}}$$

(positive sign)

Self-similar δ-plates at points $2^n a$



$$\Delta\mathcal{E}(2a) = \Delta\mathcal{E}(4a) - \frac{\pi^2}{1440(2a)^3}$$

Where we have assumed Dirichlet boundary conditions.

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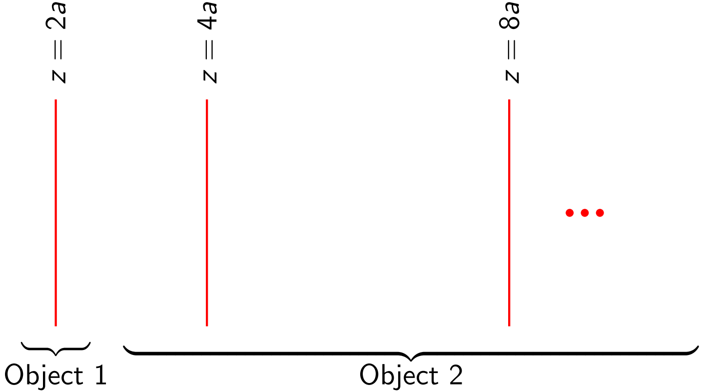
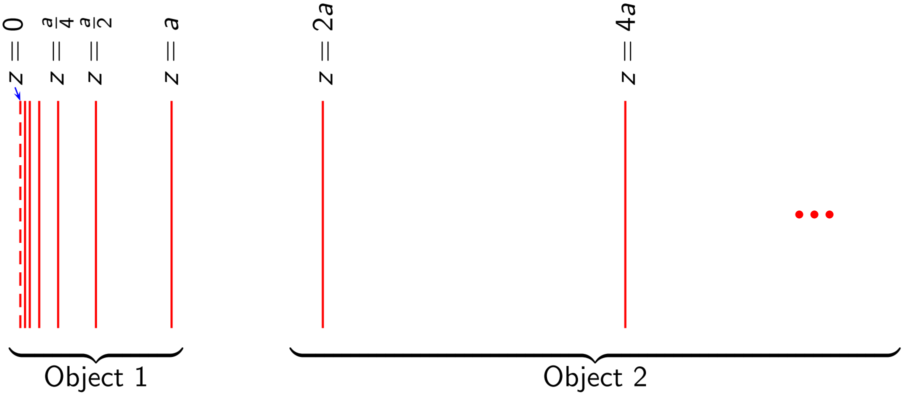


$$\Delta\mathcal{E}(2a) = \Delta\mathcal{E}(4a) - \frac{\pi^2}{1440(2a)^3}$$

Where we have assumed Dirichlet boundary conditions.
Dimensional arguments lead us to, $\Delta\mathcal{E}(4a) = \frac{1}{2^3}\Delta\mathcal{E}(2a)$.

Both sequences of plates

We consider now plates at positions $\frac{a}{2^n}$ and $2^n a$ which extend now to the whole space,



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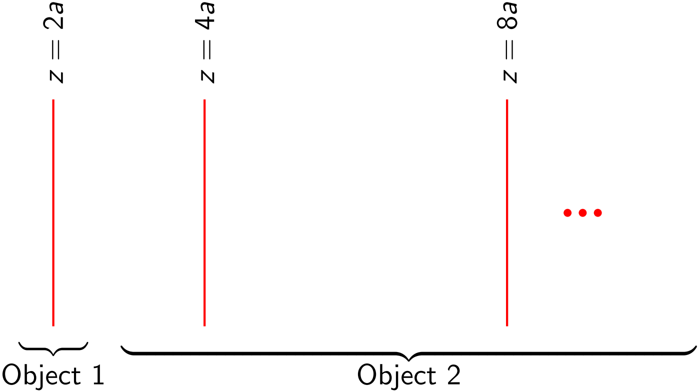

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Dimensional arguments lead us to, $\Delta\mathcal{E}(4a) = \frac{1}{2^3}\,\Delta\mathcal{E}(2a)$.
We immediately learn that

$$\boxed{\Delta\mathcal{E}(2a) = -\frac{1}{7}\frac{\pi^2}{1440a^3}} \qquad \text{(negative sign)}$$

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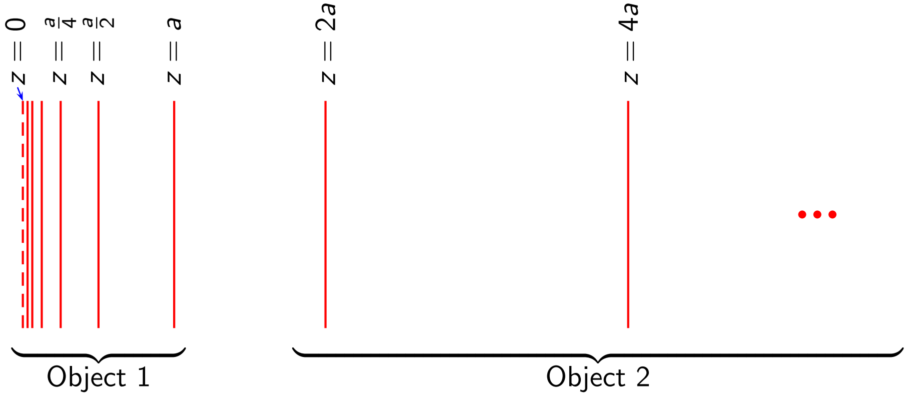

$$\Delta\mathcal{E}_{total}(a) = \Delta\mathcal{E}_{object1} + \Delta\mathcal{E}_{object2} + \Delta\mathcal{E}_{12}$$

Both sequences of plates

The interaction energy between the two objects defined above is

$$\Delta\mathcal{E}_{total}(a) = \Delta\mathcal{E}_{object1} + \Delta\mathcal{E}_{object2} + \Delta\mathcal{E}_{12}.$$

We have already calculated every term using Dirichlet b.c.,

$$\begin{aligned}
\Delta\mathcal{E}_{object1} &= \Delta\mathcal{E}(a) = +\frac{8}{7}\frac{\pi^2}{1440a^3} \\
\Delta\mathcal{E}_{object2} &= \Delta\mathcal{E}(2a) = -\frac{1}{7}\frac{\pi^2}{1440a^3} \\
\Delta\mathcal{E}_{12}(a) &= -\frac{\pi^2}{1440a^3}
\end{aligned}$$

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\end{aligned}$$

Put together we find,

$$\boxed{\Delta\mathcal{E}_{\text{tot}}(a) = +\frac{8}{7}\frac{\pi^2}{1440a^3} - \frac{1}{7}\frac{\pi^2}{1440a^3} - \frac{\pi^2}{1440a^3} = 0}$$

Both stacks of plates balance each other with opposite tendences so that together they contribute to cero.

A very 'singular' potential

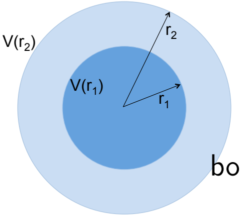
Consider a scalar massless field.

$$\mathcal{L} \equiv \frac{1}{2}(\partial_\tau \phi)^2 - \frac{1}{2}\delta^{ij}\partial_i\phi\partial_j\phi - V_{\delta\text{-}\delta'}(r)\phi^2.$$

General matching conditions of a scalar field interacting with two singular potentials with support on the boundary of two concentric spheres.

$$V_{\delta\text{-}\delta'}(r) \equiv \sum_{i=1}^{2} a_i\,\delta(r-r_i) + b_i\,\delta'(r-r_i)$$

$$a_i, b_i \in \mathbb{R}, \quad r_1 < r_2.$$



We have two interacting non intersecting bodies. We can use the TGTG formula,

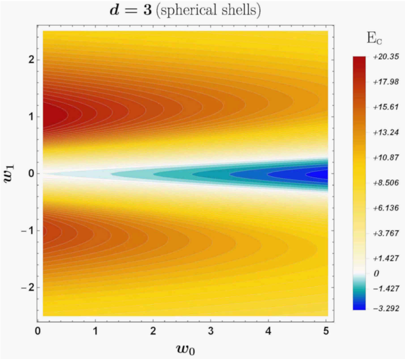
$$E_{\mathsf{C}} = \frac{\hbar}{2\pi}\int_0^\infty d\xi \,\mathrm{Tr}\,\ln\left(\mathbb{I} - \mathbb{M}(i\xi)\right), \quad \mathbb{M} = \mathbb{T}_1\mathbb{G}^0_{12}\mathbb{T}_2\mathbb{G}^0_{21}.$$

Where $\mathbb{T}_1$ and $\mathbb{T}_2$ are the transition matrices or the Lippmann-Schwinger T-operator associated to each object

Numerical results, $V = \sum_i^2 [\alpha_i \delta(x - x_i) + \beta_i \delta'(x - x_i)]$

We show results from the numerical calculation in different situations.

$$\alpha_1 = \alpha_2 = \mathrm{w}_0$$
$$\beta_1 = \beta_2 = \mathrm{w}_1$$



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$$
\begin{aligned}
\alpha_1 &= \alpha_2 = \mathrm{w}_0 \\
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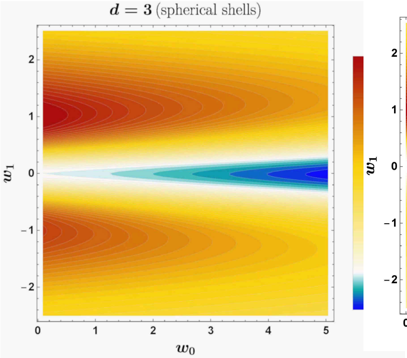
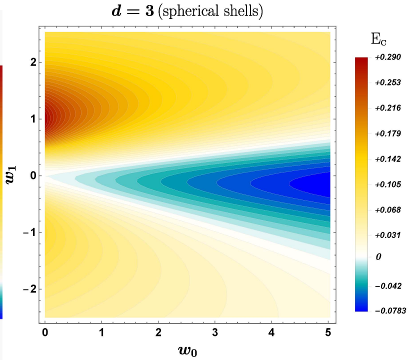
\end{aligned}
$$





Left, $x_1 = 1, 8$, and $x_2 = 2$. Right $x_1 = 1$, and $x_2 = 2$.

Self-energy of a $\delta - \delta'$ sphere

Self energies are more difficult to deal with in terms of divergences and one can not always extract a finite part.
The Casimir energy can be studied by making use of the zeta function,

$$E_0 = \frac{\mu^{2s}}{2} \sum_n \omega_n^{1-2s} = \frac{\mu^{2s}}{2} \zeta_P(s - \frac{1}{2}),$$

and evaluate it at $s \to 0$, where μ is a parameter with dimensions of mass introduced to keep the right dimensions and $\hbar = c = 1$.
The zeta function associated with the operator determining the modes of the system is

$$\zeta_P(s) = \sum_n \lambda_n^{-s}, \quad P\varphi_n(\boldsymbol{x}) = \lambda_n \varphi_n(\boldsymbol{x}).$$

that is connected to the heat kernel function $K(t)$ through the Melling transform,

$$\zeta(s) = \int_0^\infty \mathrm{d}t \, \frac{t^{s-1}}{\Gamma(s)} K(t),$$

where

where

$$K(t) := \sum_n \exp^{-\lambda_n t}.$$

The asymptotic expansion for small t is

$$K(t) \sim \frac{1}{(4\pi t)^{3/2}} \sum_n a_{n/2} t^{n/2}.$$

Having the coefficient $a_2 \neq 0$ is a guarantee that the self-energy of the system is finite.
Of course that does not imply that we find divergences in the way, which after adding and subtracting asymptotic terms, we find the expression

$$a_2 = \frac{2\pi\left(128\lambda_1^3 + 140\lambda_0^2\lambda_1 r_0^2 - 35\lambda_0^3 r_0^3 - 224\lambda_0\lambda_1^2 r_0\right)}{105\left(\lambda_1^2+1\right)^3 r_0},$$

that happens to be zero when $c_0\lambda_1 = \lambda_0 r_0, \quad c_0 \simeq 1{,}20818671192.$