SATURNALIA '24

DARK TOPICS FOR THE LONGEST NIGHTS

Phenomenology of theories of gravitation beyond general relativity

Miguel Pardina

Organisation: Mathieu Kaltschmidt, Siannah Peñaranda, Javier Redondo & Laura Seguí

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DARK TOPICS FOR THE LONGEST NIG…

Stability of cosmological singularity-free solutions in quadratic gravity

Manuel Asorey[a], Fernando Ezquerro[b] and Miguel Pardina[c]

https://arxiv.org/abs/2412.10111

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Introduction

$$S_{EH} = \frac{M_p^2}{2} \int d^4x \sqrt{-g}\, R$$

$$\Delta \mathcal{L}_{\text{1-loop}} = a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + a_3 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + a_4 \Box R$$

$$\Delta \mathcal{L}_{\text{2-loop}} \sim R \Box R,\; R^{\alpha\beta}{}_{\mu\nu} R^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta}$$

$$\Delta \mathcal{L}_{\text{3-loop}} \sim O(R^4)$$

$$\vdots$$

Fixed a_i coeficients

GR is not renormalizable!

Introduction

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$$\Delta \mathcal{L}_{\text{3-loop}} \sim O(R^4)$$

⋮

Fixed a_i coeficients

Non-renormalizable

EFT perspective?

Introduction

$$S_{EH} = \frac{M_p^2}{2}\int d^4x\sqrt{-g}\,R$$

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$$\Delta\mathcal{L}_{\text{3-loop}} \sim O(R^4)$$

~~Fixed a_i coeficients~~

⋮

GR is not renormalizable!

Introduction

$$S_{EH} = \frac{M_p^2}{2} \int d^4x \sqrt{-g}\, R$$

$$\gamma = \frac{M_p^2}{2}$$

Modified with free parameters

$$S = \int d^4x\, \mathcal{L} = \int d^4x \sqrt{-g}\left(\gamma R + \alpha R^2 - \beta R_{\mu\nu} R^{\mu\nu}\right)$$

Quadratic Gravity

Renormalizable!

Differences with respect to GR?

Cosmological solutions?

Inflation?

Higher derivative gravity

Stelle's theory $$S = \int d^4x \sqrt{-g}\left(\gamma R + \alpha R^2 - \beta R_{\mu\nu}R^{\mu\nu}\right)$$

Quadratic terms as $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ or $\Box R$?

Gauss-Bonnet in $D = 4$ is a total derivative

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$$

$\Box R$ is also a pure derivative

$$\sqrt{-g}\Box R = \sqrt{-g}\nabla_\mu \nabla^\mu R = \partial_\mu\left(\sqrt{-g}\nabla^\mu R\right)$$

Ostrogradsky instability and ghosts

Quadratic terms → Eqs with higher derivatives

Clasically unstable!

Ghosts in the quantum theory!

$$S = \int d^4x\sqrt{-g}\left(\gamma R + \alpha R^2 - \beta R_{\mu\nu}R^{\mu\nu}\right)$$

Graviton

$$s = 2, m = 0$$

Scalaron

$$s_0 = 0, m_0 = \sqrt{\frac{\gamma}{2(3\alpha - \beta)}}$$

Ghost

$$s_2 = 2, m_2 = \sqrt{\frac{\gamma}{\beta}}$$

Ostrogradsky instability and ghosts

Quadratic terms → Eqs with higher derivatives

Clasically unstable!

Ghosts in the quantum theory!

$$S = \int d^4x\sqrt{-g}\left(\gamma R + \alpha R^2 - \beta R_{\mu\nu}R^{\mu\nu}\right)$$

Decoupling (explicitly or EFT)

Ficticious by finite derivatives

Pole integrating contours

PT symmetric Hamiltonian

Ghost

$s_2 = 2, m_2 = \sqrt{\frac{\gamma}{\beta}}$

Cosmological solutions in quadratic gravity

$$S = \int d^4x \sqrt{-g}\left[\gamma(R-2\Lambda) + \alpha R^2 - \beta R_{\mu\nu}R^{\mu\nu}\right] + S_m$$

$$\gamma\left(G_{\mu\nu} + \Lambda g_{\mu\nu}\right) + \Phi_{\mu\nu} = \frac{1}{2}T_{\mu\nu}$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} ,$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R ,$$

$$\Phi_{\mu\nu} = \alpha\left(2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2 - 2\nabla_\mu\nabla_\nu R + 2g_{\mu\nu}\Box R\right)$$
$$- \beta\left(-\frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - \nabla_\nu\nabla_\mu R + \Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Box R - 2R_{\alpha\mu\nu\beta}R^{\alpha\beta}\right)$$

Cosmological solutions in quadratic gravity

$$\gamma \left(G_{\mu\nu} + \Lambda g_{\mu\nu}\right) + \Phi_{\mu\nu} = \frac{1}{2} T_{\mu\nu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,,$$

$$\Phi_{\mu\nu} = \alpha \left(2RR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^2 - 2\nabla_\mu \nabla_\nu R + 2 g_{\mu\nu} \Box R\right)$$

$$- \beta \left(-\frac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - \nabla_\nu \nabla_\mu R + \Box R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \Box R - 2 R_{\alpha\mu\nu\beta} R^{\alpha\beta}\right)$$

$\Lambda = 0$ & $T_{\mu\nu} = 0$, every sol of $G_{\mu\nu} = 0$ is also sol.

$\Lambda \neq 0$ & $T_{\mu\nu} = 0$, maximally symmetric spaces are sol only in $D = 4$.

Sols of the complete theory are usually non isotropic.

Field equations in a FLRW metric

$$ds^2 = -b(t)^2 dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)\ , \quad g_{\mu\nu} = \mathrm{diag}(-b^2, a^2, a^2, a^2)$$

$$R = \frac{6}{b^2}(\Omega + \Theta)\ , \quad R^2 = \frac{36}{b^4}\left(\Omega^2 + \Theta^2 + 2\Omega\Theta\right)\ , \quad R_{\mu\nu}R^{\mu\nu} = \frac{12}{b^4}\left(\Omega^2 + \Theta^2 + \Omega\Theta\right)\ ,$$

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{12}{b^4}(\Omega^2 + \Theta^2)\ , \quad G = \frac{24}{b^4}\Omega\Theta\ , \quad \text{where} \;\; \Omega = \frac{\ddot{a}}{a} - \frac{\dot{a}}{a}\frac{\dot{b}}{b}\ , \quad \Theta = \frac{\dot{a}^2}{a^2}\ ,$$

$$S = \int d^4x \sqrt{-g}\left[\gamma(R - 2\Lambda) + \alpha R^2 - \beta R_{\mu\nu}R^{\mu\nu}\right] + S_m$$

$$S = S_g + S_m = \int d^3x \int dt\, \mathcal{L}(a, \dot{a}, \ddot{a}, b, \dot{b}) + S_m$$

$$\delta S_g = -\delta S_m$$

Field equations in a FLRW metric

$$\mathcal{E}_b\left(a,\dot{a},\ddot{a},\dddot{a},b,\dot{b},\ddot{b}\right)=\frac{\partial\mathcal{L}}{\partial b}-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{b}}\right)=a^3\rho\ ,$$

$$\mathcal{E}_a\left(a,\dot{a},\ddot{a},\dddot{a},\ddddot{a},b,\dot{b},\ddot{b},\dddot{b}\right)=\frac{\partial\mathcal{L}}{\partial a}-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{a}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial\mathcal{L}}{\partial\ddot{a}}\right)=-3a^2bp\ .$$

Not independent! $\dot{\mathcal{E}}_b=\frac{\dot{a}}{b}\mathcal{E}_a$

$$\left.\begin{aligned}A\ddddot{a}+B\dddot{b}&=P\\C\ddddot{a}+D\dddot{b}&=Q\end{aligned}\right\}\begin{pmatrix}A&B\\C&D\end{pmatrix}\begin{pmatrix}\ddddot{a}\\\dddot{b}\end{pmatrix}=\begin{pmatrix}P\\Q\end{pmatrix}$$

Non invertible matrix!

Infinite sols, only one solution if we fix the gauge

$$b(t)=1$$

Field equations in a FLRW metric

$$\frac{12(3\alpha-\beta)\dot{a}^2\left(2a\ddot{a}-3\dot{a}^2\right)}{a}+6a\left[\gamma\dot{a}^2-2(3\alpha-\beta)\left(\ddot{a}^2-2\,\dddot{a}\,\dot{a}\right)\right]-2\gamma\Lambda a^3=a^3\rho$$

$$6\left[\gamma\left(\dot{a}^2-\Lambda a^2\right)+\frac{6(3\alpha-\beta)\left(\dot{a}^4-4a\dot{a}^2\ddot{a}\right)}{a^2}+2a\left(2(3\alpha-\beta)\,\ddddot{a}+\gamma\ddot{a}\right)\right.$$
$$\left.+2(3\alpha-\beta)\left(3\ddot{a}^2+4\,\dddot{a}\,\dot{a}\right)\right]=-3a^2p\,.$$

There's one sol & we only need one eq (linked by matter)

If $3\alpha=\beta$ we recover Friedmann eqs:

$$6a\gamma\dot{a}^2-2\gamma\Lambda a^3=a^3\rho\quad\Longrightarrow\quad\frac{\dot{a}^2}{a^2}=\frac{\rho}{6\gamma}+\frac{\Lambda}{3}\,,$$

$$6\left[\gamma\left(\dot{a}^2-\Lambda a^2\right)+2a\gamma\ddot{a}\right]=-3a^2p\quad\Longrightarrow\quad\frac{\ddot{a}}{a}=-\frac{1}{12\gamma}\left(\rho+3p\right)+\frac{\Lambda}{3}$$

Field equations in a FLRW metric

Why? $$R_{\mu\nu}R^{\mu\nu} = \frac{1}{3}R^2 + \frac{1}{2}W^2 - \frac{1}{2}G$$

In FRW like metrics it is verified $W^2 = 0$. Therefore:

$$S = \int d^4x\sqrt{-g}\left(\gamma R + \alpha R^2 - \beta R_{\mu\nu}R^{\mu\nu}\right)$$

↓ Equivalent to

$$S = \int d^4x\sqrt{-g}\left[\gamma R + \left(\alpha - \frac{\beta}{3}\right)R^2 - \frac{\beta}{2}W^2 + \frac{\beta}{2}G\right]$$

↓ In FLRW metrics

Starobinsky! $$S = \int d^4x\sqrt{-g}\left[\gamma R + \left(\alpha - \frac{\beta}{3}\right)R^2\right]$$

de Sitter solution, stability & future of the universe

Field equations ($\rho = 0$, $H = \dot{a}/a$):

$$3\gamma H^2 - \gamma\Lambda + 6(3\alpha - \beta)\left(-\dot{H}^2 + 6\dot{H}H^2 + 2H\ddot{H}\right) = 0$$

de Sitter space with $H = H_0 = \sqrt{\Lambda/3}$ is a sol.

It is stable if and only if $3\alpha > \beta$.

de Sitter solution, stability & future of the universe

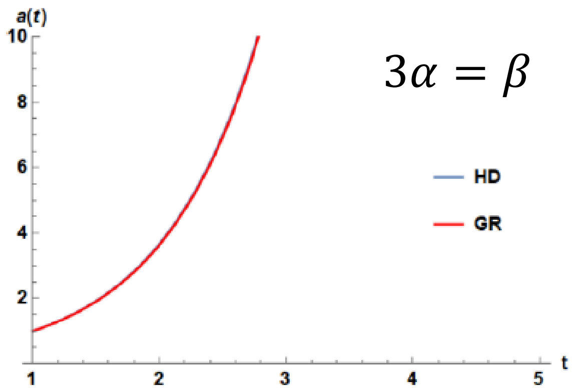


Figura 1: $\Lambda = 5$, $\alpha = 1$, $\beta = 2.9999$, $\gamma = 20$, $a(1) = 1$, $\dot{a}(1) = 1$, $\ddot{a}(1) = 0.2$.

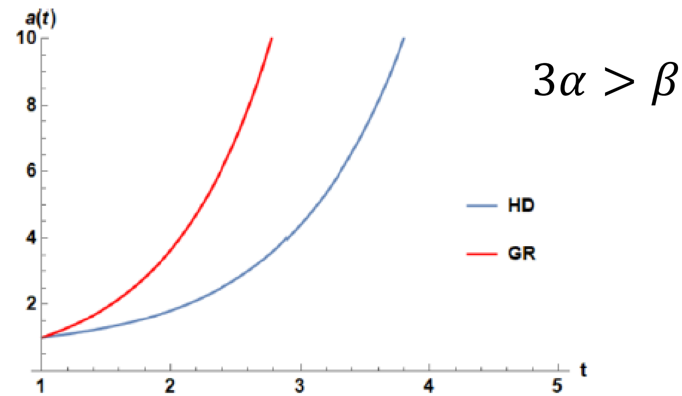


Figura 2: $\Lambda = 5$, $\alpha = 5$, $\beta = 1$, $\gamma = 20$, $a(1) = 1$, $\dot{a}(1) = 0.5$, $\ddot{a}(1) = 0.3$.

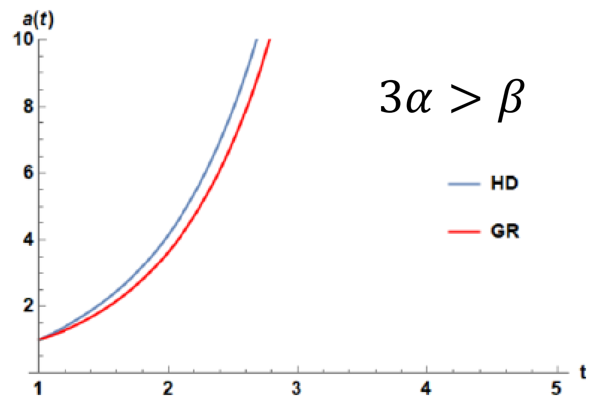


Figura 3: $\Lambda = 5$, $\alpha = 5$, $\beta = 1$, $\gamma = 20$, $a(1) = 1$, $\dot{a}(1) = 1.9$, $\ddot{a}(1) = 0.5$.

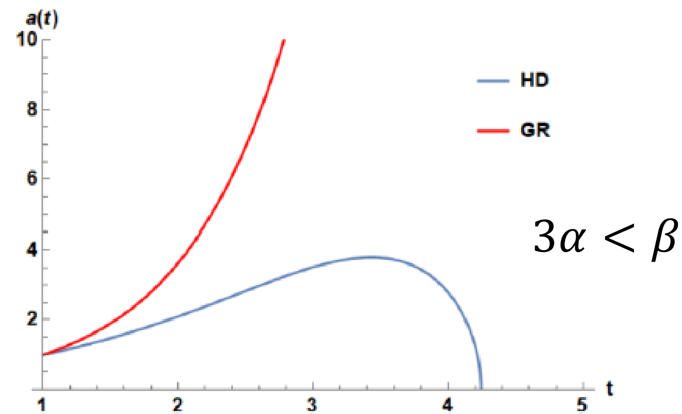
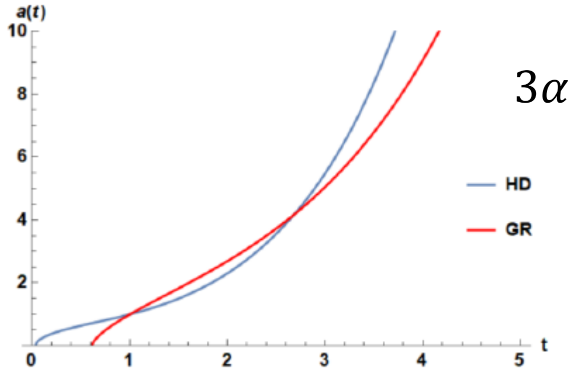


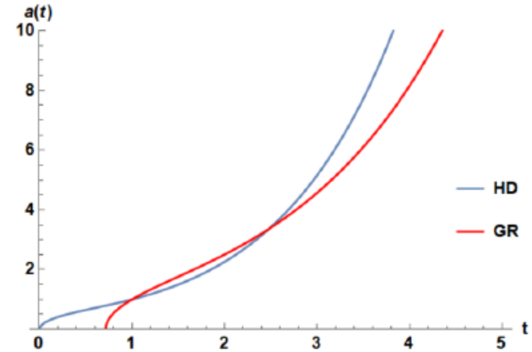
Figura 4: $\Lambda = 5$, $\alpha = 5$, $\beta = 30$, $\gamma = 20$, $a(1) = 1$, $\dot{a}(1) = 0.8$, $\ddot{a}(1) = 0.6$.

Past of the universe & initial singularity



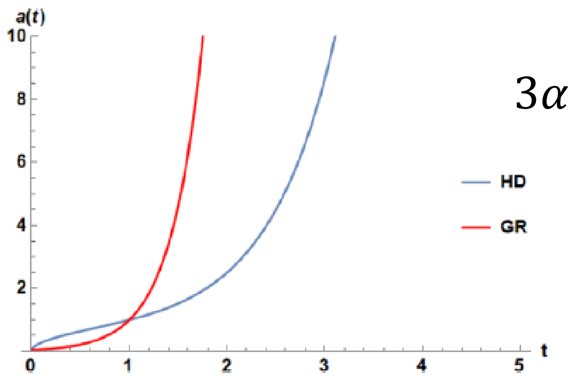
$3\alpha > \beta$

Figura 5: $\omega = 0$, $\rho_0 = 170$, $\Lambda = 1$, $\alpha = 4$, $\beta = 3$, $\gamma = 10$, $a(1) = 1$, $\dot{a}(1) = 0.8$, $\ddot{a}(1) = 0.6$.



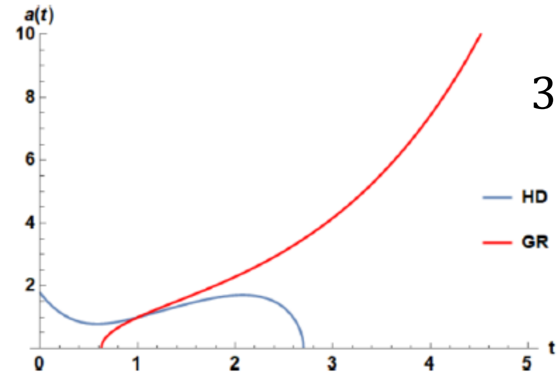
$3\alpha > \beta$

Figura 6: $\omega = 1/3$, $\rho_0 = 180$, $\Lambda = 1$, $\alpha = 6$, $\beta = 3$, $\gamma = 10$, $a(1) = 1$, $\dot{a}(1) = 0.8$, $\ddot{a}(1) = 0.6$.



$3\alpha > \beta$

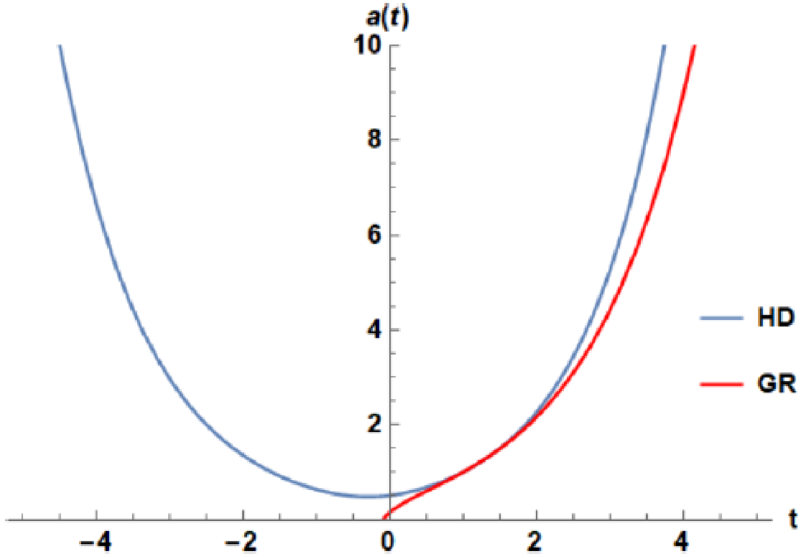
Figura 7: $\omega = -1$, $\rho_0 = 540$, $\Lambda = 1$, $\alpha = 8$, $\beta = 4$, $\gamma = 10$, $a(1) = 1$, $\dot{a}(1) = 0.8$, $\ddot{a}(1) = 0.6$.



$3\alpha < \beta$

Figura 8: $\omega = 1/3$, $\rho_0 = 100$, $\Lambda = 1$, $\alpha = 5$, $\beta = 16$, $\gamma = 10$, $a(1) = 1$, $\dot{a}(1) = 0.8$, $\ddot{a}(1) = 0.6$.

Singularity-free solution



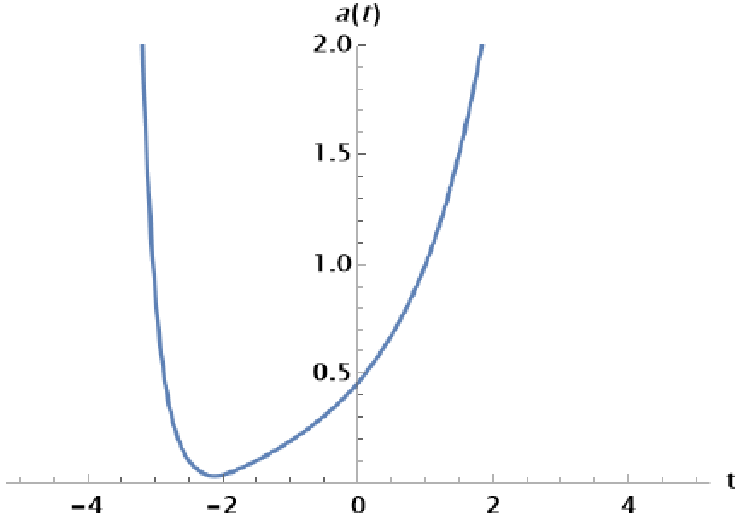
$3\alpha < \beta$

FIG. 9. Singularity-free solution of motion equations corresponding to the choice of parameters: $\omega = 0$, $\rho_0 = 15$, $\Lambda = 1.5$, $\alpha = 3$, $\beta = 16$, $\gamma = 10$, $a(1) = 1$, $\dot{a}(1) = 0.8$, $\ddot{a}(1) = 0.7$.

Stability of the nonsingular solutions

$$24\begin{pmatrix} \frac{\gamma}{2}\ddot{a}+\frac{(3\alpha-\beta)\dot{a}^4}{a^3}-\frac{\rho_0\omega(3\omega+1)}{8}a^{-3\omega-2}-\frac{\gamma}{2}\Lambda a & \dot{a}\left(\frac{2(\beta-3\alpha)\dot{a}^2}{a^2}+\frac{\gamma}{2}\right) & (3\alpha-\beta)\ddot{a}+\frac{\gamma}{2}a \\ \dot{a}\left(\frac{2(\beta-3\alpha)\dot{a}^2}{a^2}+\frac{\gamma}{2}\right) & (6\alpha-\beta)\ddot{a}+\frac{6(3\alpha-\beta)\dot{a}^2}{a}+\frac{\gamma}{2}a & (6\alpha-\beta)\dot{a} \\ (3\alpha-\beta)\ddot{a}+\frac{\gamma}{2}a & (6\alpha-\beta)\dot{a} & (3\alpha-\beta)a \end{pmatrix}$$

Quadratic form
Hessian



$$t_c \approx -2.12$$

$$a(t_c) \approx 0.028$$

FIG. 10. Nonsingular solution for parameters: $\omega = 0$, $\rho_0 = 1.66$, $\Lambda = 0.5$, $\alpha = 4$, $\beta = 19$, $\gamma = 10$, $a(1) = 1$, $\dot{a}(1) = 0.8$, $\ddot{a}(1) = 0.7$.

Stability of the nonsingular solutions

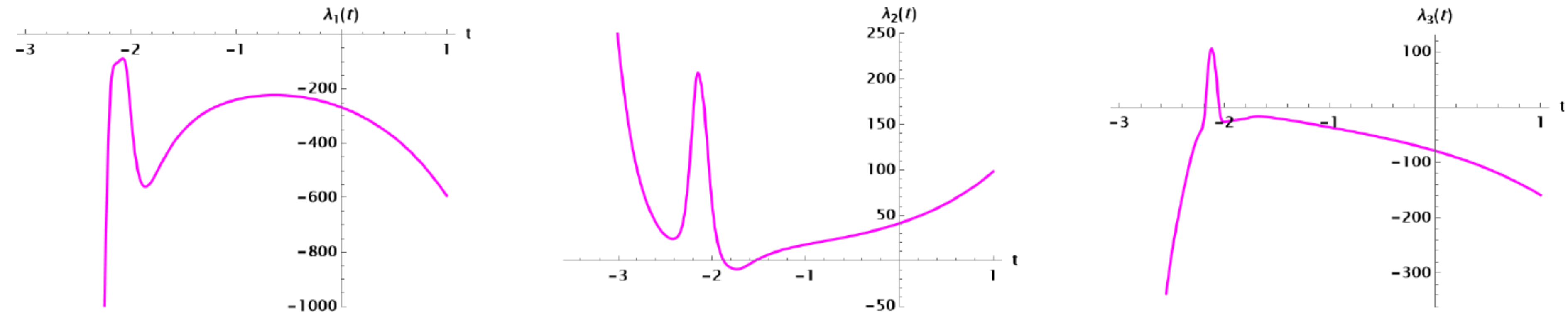


FIG. 11. Eigenvalues $\lambda_1(t), \lambda_2(t), \lambda_3(t)$ associated to the second order variation of the action for the solution shown in Figure 10.

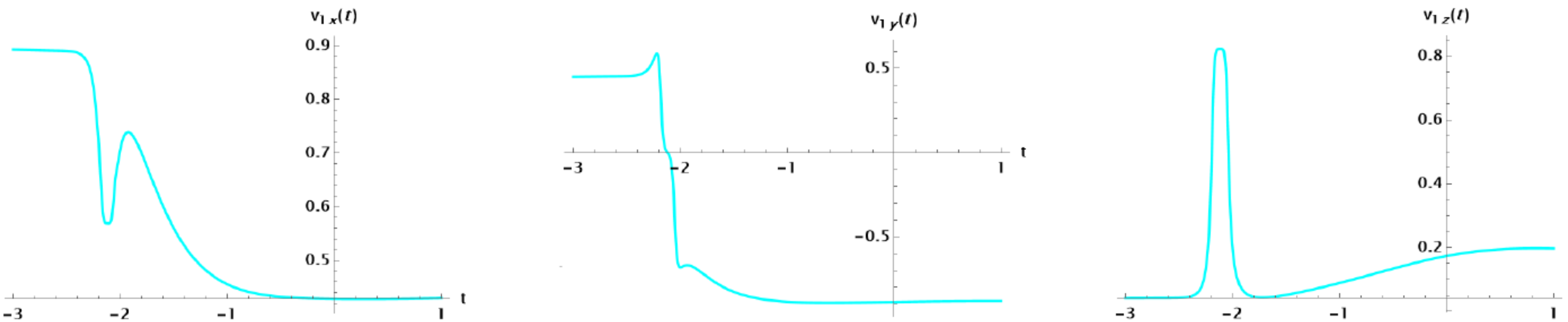


FIG. 12. Components $v_{1x}(t), v_{1y}(t), v_{1z}(t)$ of the eigenvector associated to the largest unstable eigenvalue $\lambda_1(t)$.

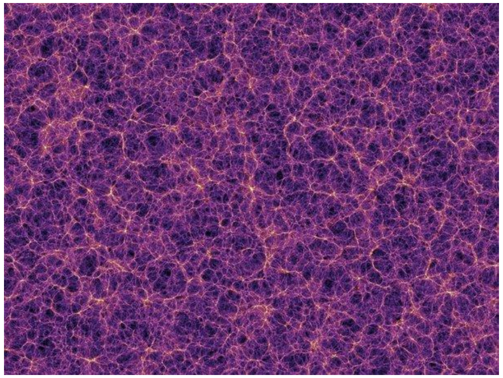
$$x, y, z \longleftrightarrow a, \dot{a}, \ddot{a}$$

Inflationary paradigm basics

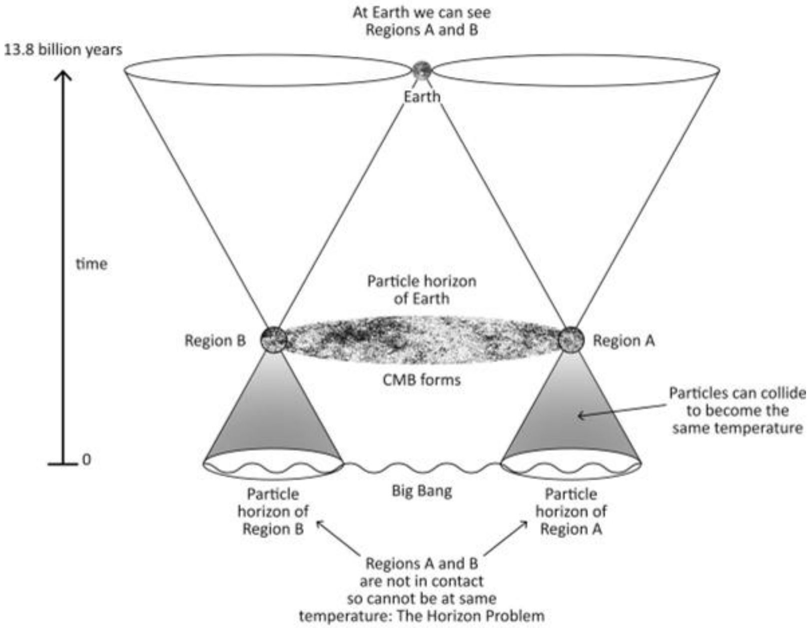
ΛCDM: flat, homogenous & isotropic universe.

Very robust cosmological model

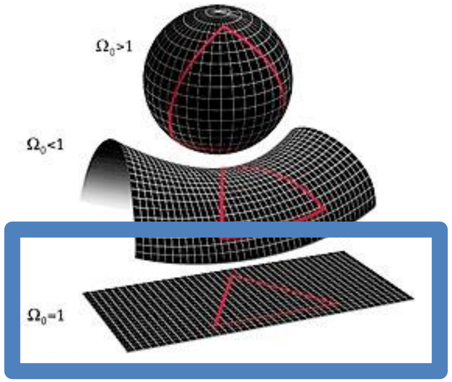
However… 3 non solved puzzles



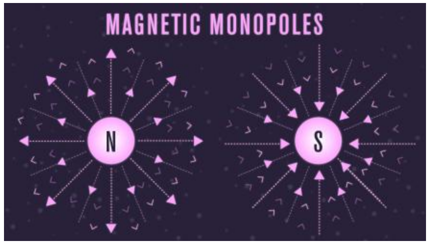
1. Horizon problem



2. Flatness problem



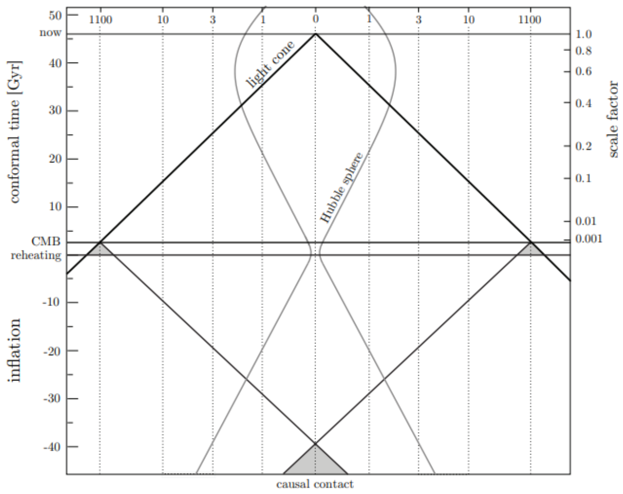
3. Exotic relics



Inflationary paradigm basics

Inflation mechanism explains all three of them

1. Horizon problem



2. Flatness problem

$$(\Omega^{-1} - 1)\rho a^2 = \frac{-3kc^2}{8\pi G},$$

$$a(t) \propto \exp(H_0 t)$$

Right hand side is cte, curvature has to decrease.

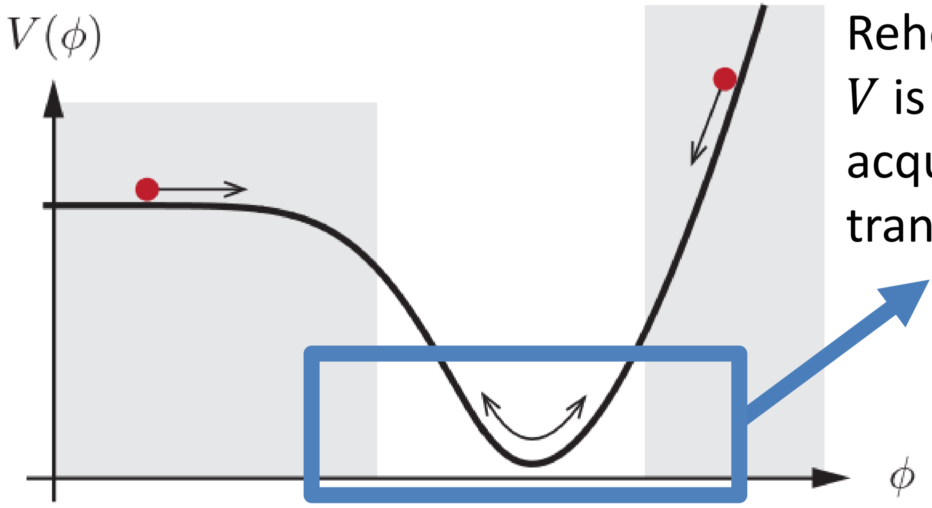
3. Exotic relics

Created before/ during inflation: density of 1 per Hubble volume, the expansion dilutes them.

Inflation requisites: almost de Sitter inflation, H almost constant (slow-roll), $p < 0$ & $\rho \sim cte$.

Inflationary paradigm basics

Usually described via a homogenous scalar field $\phi(t)$



Reheating (inflation ends) V is high & the field acquires energy, which is transfered to the SM.

Inflation parameters:

$$0 < \epsilon = -\frac{\dot{H}}{H^2}$$

$$\mathcal{N} \equiv \log\left(\frac{a_f}{a_i}\right) = \int_{t_i}^{t_f} dt H(t) \simeq 3\int_{\phi_f}^{\phi_i} d\phi \frac{H^2}{V'(\phi)} \qquad \eta = -\frac{\dot{H}}{H^2} - \frac{\ddot{H}}{2H\dot{H}}$$

$$A_s = \frac{H^2}{16\pi^2\gamma\epsilon} = \frac{H^2}{8\pi^2 M_p^2 \epsilon}\,, \qquad n_s = 1 - 6\epsilon + 2\eta\,, \qquad r = 16\epsilon$$

Starobinsky inflationary model

$$S = \int d^4x \sqrt{-g}\, \gamma \left(R + \frac{\alpha}{\gamma} R^2 \right) , \qquad \frac{\alpha}{\gamma} = \frac{1}{6M^2}$$

Conformal transformation

$$g_{\mu\nu} \;\to\; \tilde{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu} \equiv e^{2\omega(x)} g_{\mu\nu}$$

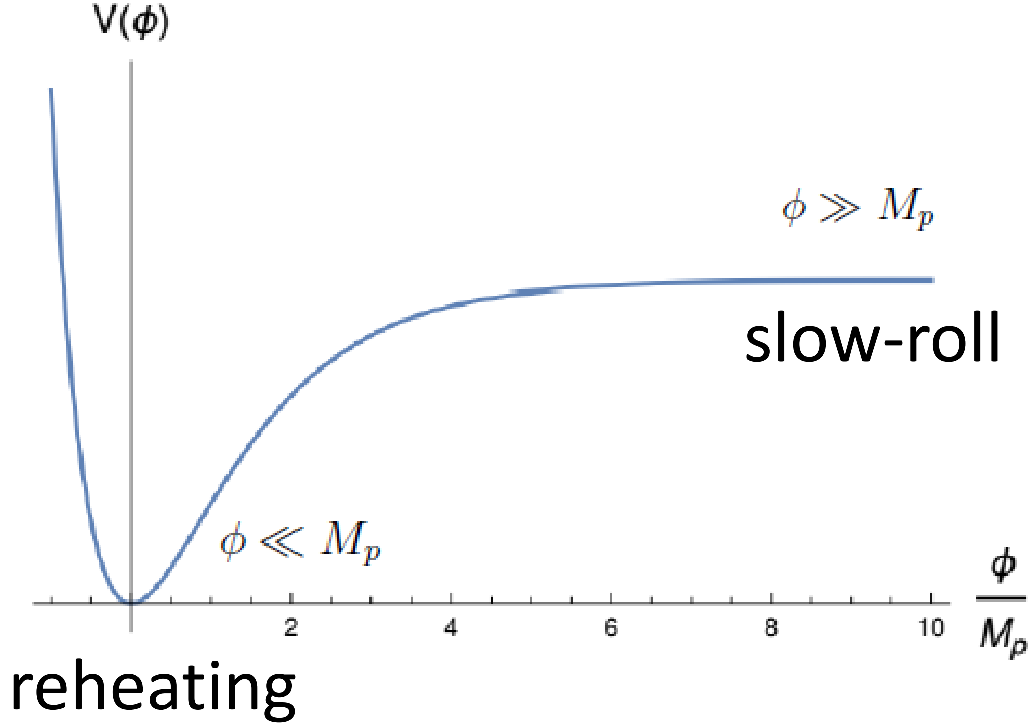
Starobinsky = GR + coupled scalar field

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\gamma \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

$$\phi(x) = \sqrt{12\gamma}\, \omega(x) \qquad V(\phi) = \frac{\gamma^2}{4\alpha} (1 - e^{-2\omega})^2 = \frac{3}{4} M_p^2 M^2 \left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\phi}{M_p}} \right)^2$$

Starobinsky inflationary model

$$V(\phi) = \frac{3}{4} M_p^2 M^2 \left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_p}}\right)^2$$



Starobinsky inflationary model

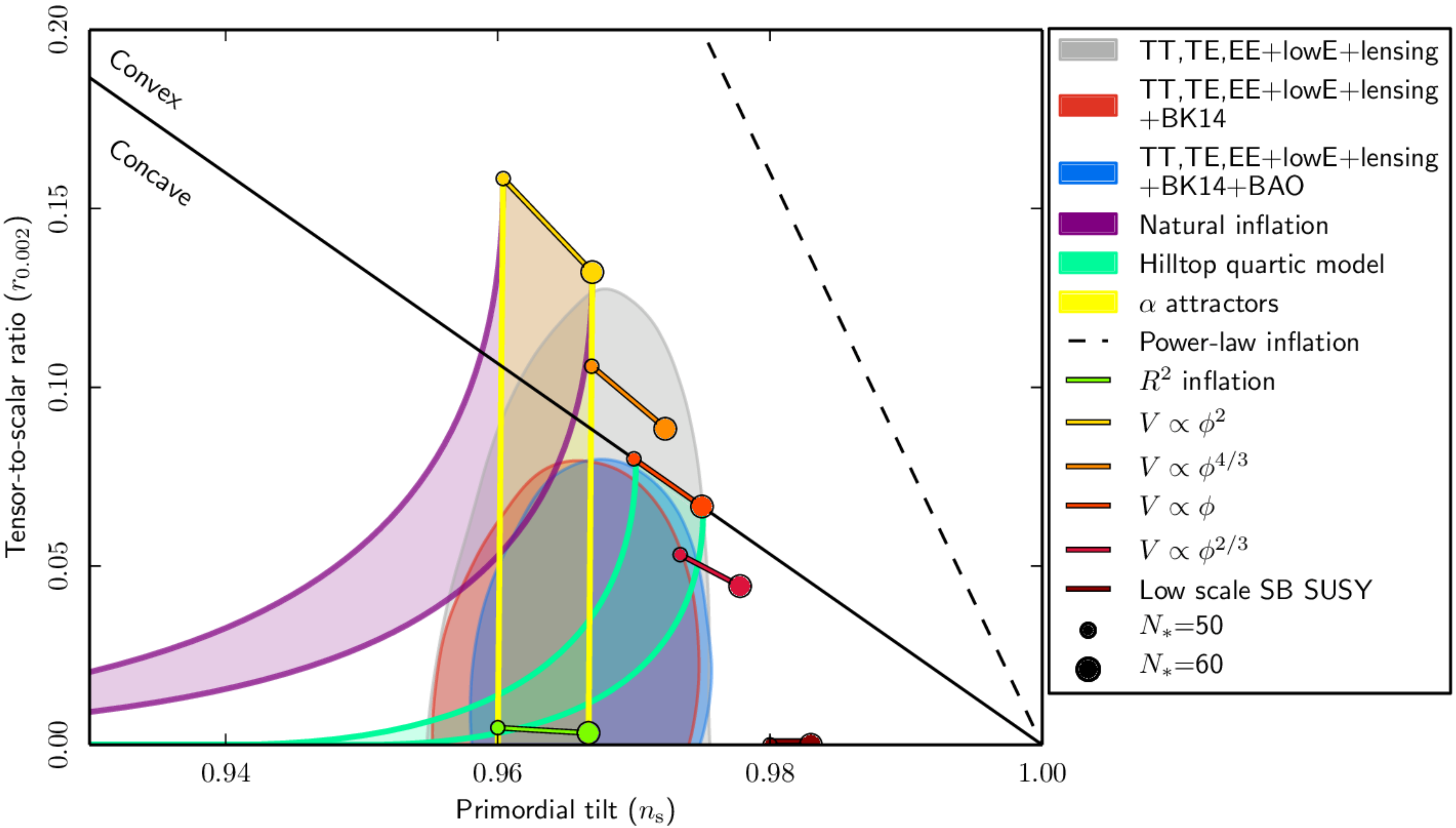
$$\epsilon = \frac{M_p^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = \frac{4}{3}\left(1 - e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_p}}\right)^{-2} , \quad \eta = M_p^2 \frac{V''(\phi)}{V(\phi)} = -\frac{4}{3}\frac{\left(e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_p}} - 2\right)}{\left(1 - e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_p}}\right)^2} ,$$

$$\mathcal{N}(\phi) = \frac{1}{M_p^2}\int_{\phi}^{\phi_i} d\phi \frac{V(\phi)}{V'(\phi)} \simeq \frac{3}{4}\left(e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_p}} + \sqrt{\frac{2}{3}}\frac{\phi}{M_p}\right)$$

$$\epsilon \simeq \frac{3}{4\mathcal{N}^2} , \qquad \eta \simeq \frac{1}{\mathcal{N}} ,$$

$$n_s \simeq 1 - \frac{2}{\mathcal{N}} , \qquad r \simeq \frac{12}{\mathcal{N}^2}$$

Starobinsky inflationary model



Quadratic gravity model

$$S=\int d^4x\sqrt{-g}\left(\gamma R+\alpha R^2-\beta R_{\mu\nu}R^{\mu\nu}\right)$$

FLRW metrics

$$S=\int d^4x\sqrt{-g}\left[\gamma R+\left(\alpha-\frac{\beta}{3}\right)R^2\right]$$

Same predictions of Starobinsky! $\left(\alpha-\frac{\beta}{3}\right)\leftrightarrow\frac{\gamma}{6M^2}$

$$S=\int d^4x\sqrt{-g}\,\gamma\left(R+\frac{\alpha}{\gamma}R^2\right)\,,\qquad \frac{\alpha}{\gamma}=\frac{1}{6M^2}$$

Conclusions

Quadratic gravity:

✓ Renormalizable and UV-completes GR.

X Unitarity, causality, ghosts, etc.

- New cosmological solutions, some of them singularity-free.

✓ Best phenomenological predictions of the inflationary era.