Universidad de Burgos
Mathematical Physics Group

CAPA

Q-CAYLE
COMUNICACIONES CUÁNTICAS

DOUBLY SPECIAL RELATIVITY AS A NON-LOCAL QUANTUM FIELD THEORY

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CONTENTS

1. INTRODUCTION

Quantum Gravity Theories

- Attempts of **unification**: string theory, loop quantum gravity, supergravity, causal set theory...
- In most of them a **minimal length** appears $\Longrightarrow$ Planck length (l_P)??
- This is closely related to an **energy scale** $\Longrightarrow$ Planck energy (Λ)??
- Problem: there are **no experimental evidences** of a fundamental QGT
- Changing the notion of **spacetime**
  1. Classical spacetime → "quantum" spacetime
  2. Symmetries? → LI should be broken/deformed at Planckian scales
  3. New effects → Micro black holes creation?

1. INTRODUCTION

GR is perturbatively **non** renormalizable

↓

Quantum gravity approaches

1. $f(R)$ theories introducing **Ricci scalar** terms in action but **non** renomalizable

2. Introducing **squared Ricci and Riemann tensor** terms in action but $\mathcal{H}$ **unbounded** from below

3. Non-local QFT and **infinite derivative gravity**

4. **Modifying kinematics** of SR by introducing Λ
   - LIV: breaking Lorentz invariance
   - **DSR**: deformed Poincaré invariance. Also **non-local**

1. INTRODUCTION

Non-local QFT

- These theories are ghost free
- This possibility was considered in string and causal set theories
- When applied to gravity, singularities disappear

Doubly Special Relativity (DSR)

- Kinematics of SR are deformed by including a **high-energy scale** Λ
- Deformed dispersion relation

$$C(k) = k_0^2 - \vec{k}^2 + \frac{k_0^3}{\Lambda} + \ldots = m^2$$

- Deformed conservation laws (composition law of momenta)

$$\text{Total momentum} = (p \oplus q)_\mu = p_\mu + q_\mu + \frac{p_\mu q_0}{\Lambda} + \ldots$$

- Dispersion relation and conservation law compatible with **relativity principle** $\rightarrow$ deformed Lorentz transformations

1. INTRODUCTION

RELATIVE LOCALITY

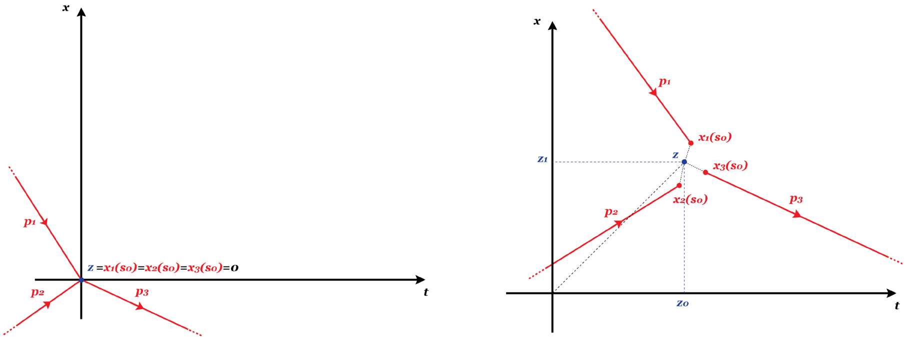
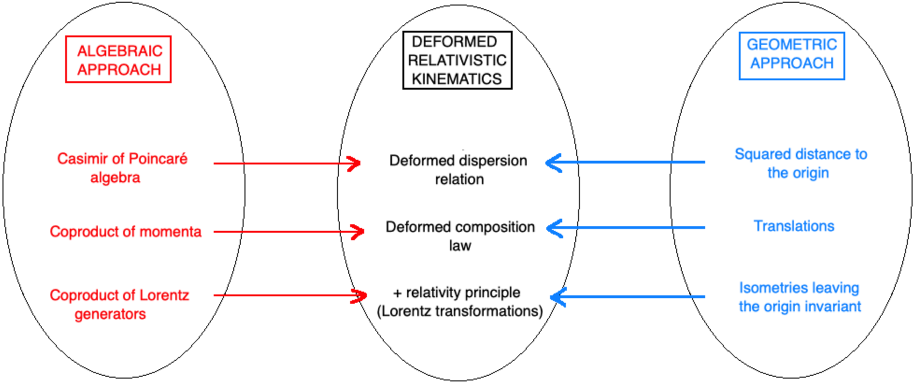


Fig.: Figures designed by Dr. Flavio Mercati

2. PREVIOUS WORKS

IDEA: If a curved spacetime describes General Relativity, maybe a maximally symmetric curved momentum space represents Quantum Gravity



2. PREVIOUS WORKS

Deformed kinematics from geometric elements (tetrad, isometries, distance)

- Starting with a **maximally symmetric momentum metric** $g_{\mu\nu}(p)$
- Computing the **Casimir** by using

$$C(p) = f^{\mu}(p) g_{\mu\nu}(p) f^{\nu}(p)\,, \qquad f^{\mu}(p) := \frac{1}{2}\frac{\partial C(p)}{\partial p_{\mu}}$$

- Computing the **composition law** by using

$$g_{\mu\nu}\left(p \oplus q\right) = \frac{\partial \left(p \oplus q\right)_{\mu}}{\partial q_{\rho}} g_{\rho\sigma}(q) \frac{\partial \left(p \oplus q\right)_{\nu}}{\partial q_{\sigma}}$$

- The composition law defines a **tetrad**

$$e^{\mu}{}_{\nu}(p) := \left.\frac{\partial \left(p \oplus q\right)_{\nu}}{\partial q_{\mu}}\right|_{q\to 0}$$

[1] J.M. Carmona, J.L. Cortés and J.J Relancio. *Phys. Rev. D 100* (2019)
[2] J.J. Relancio and S. Liberati. *Phys. Rev. D 101* (2020)

2. PREVIOUS WORKS

Different kinematics from the same metric

- *Particular example*:
  1. De Sitter metric
  $$g_{\mu\nu}(p) = \eta_{\mu\nu} + p_\mu p_\nu/\Lambda^2$$
  2. **Snyder** kinematic's isometry generators[3]
  $$\mathcal{T}_S^\lambda = \sqrt{1+\frac{\bar{p}^2}{\Lambda^2}}\,\frac{\partial}{\partial p_\lambda}\,, \qquad \mathcal{J}^{\mu\nu} = p_\rho(\delta^\nu_\lambda \eta^{\mu\rho} - \delta^\mu_\lambda \eta^{\nu\rho})\frac{\partial}{\partial p_\lambda}\,,$$
  satisfying
  $$[\mathcal{T}_S^\alpha, \mathcal{T}_S^\beta] = \frac{\mathcal{J}^{\alpha\beta}}{\Lambda^2}\,, \qquad [\mathcal{T}_S^\alpha, \mathcal{J}^{\beta\gamma}] = \eta^{\alpha\beta}\mathcal{T}_S^\gamma - \eta^{\alpha\gamma}\mathcal{T}_S^\beta\,,$$
  $$[\mathcal{J}^{\alpha\beta}, \mathcal{J}^{\gamma\delta}] = \eta^{\beta\gamma}\mathcal{J}^{\alpha\delta} - \eta^{\alpha\gamma}\mathcal{J}^{\beta\delta} - \eta^{\beta\delta}\mathcal{J}^{\alpha\gamma} + \eta^{\alpha\delta}\mathcal{J}^{\beta\gamma}$$
  3. Deformed composition law
  $$(p\oplus q)^S_\mu = p_\mu\left(\sqrt{1+\frac{q^2}{\Lambda^2}} + \frac{p_\mu \eta^{\mu\nu} q_\nu}{\Lambda^2\left(1+\sqrt{1+p^2/\Lambda^2}\right)}\right) + q_\mu$$
  4. **Noncommutativity** of the space-time coordinates $\rightarrow [x^\mu, x^\nu] = i\mathcal{J}^{\mu\nu}/\Lambda$

[3] M.V. Battisti and S. Meljanac. *Phys. Rev. D 82* (2010)

2. PREVIOUS WORKS

Different kinematics from the same metric

- *Particular example*:
  1. De Sitter metric
$$g_{\mu\nu}(p) \;=\; \eta_{\mu\nu} + p_\mu p_\nu/\Lambda^2$$
  2. **κ-Poincaré** kinematic's isometry generators[4]
$$\mathcal{T}^\mu_\kappa = \mathcal{T}^\mu_S + n_\alpha \frac{\mathcal{J}^{\mu\alpha}}{\Lambda}\,, \qquad \mathcal{J}^{\mu\nu} = p_\rho(\delta^\nu_\lambda \eta^{\mu\rho} - \delta^\mu_\lambda \eta^{\nu\rho})\frac{\partial}{\partial p_\lambda}\,,$$
     satisfying ($n_\mu := (1,0,0,0)$)
$$[\mathcal{T}^\alpha_\kappa, \mathcal{T}^\beta_\kappa] = \frac{n_\gamma}{\Lambda}\left(\mathcal{T}^\alpha_\kappa \eta^{\beta\gamma} - \mathcal{T}^\beta_\kappa \eta^{\alpha\gamma}\right)\,,$$
$$[\mathcal{T}^\alpha_S, \mathcal{J}^{\beta\gamma}] = \eta^{\alpha\beta}\mathcal{T}^\gamma_\kappa - \eta^{\alpha\gamma}\mathcal{T}^\beta_\kappa + \frac{n_\delta}{\Lambda}\left(\eta^{\delta\beta}\mathcal{J}^{\alpha\gamma} - \eta^{\delta\gamma}\mathcal{J}^{\alpha\beta}\right)$$
$$[\mathcal{J}^{\alpha\beta}, \mathcal{J}^{\gamma\delta}] = \eta^{\beta\gamma}\mathcal{J}^{\alpha\delta} - \eta^{\alpha\gamma}\mathcal{J}^{\beta\delta} - \eta^{\beta\delta}\mathcal{J}^{\alpha\gamma} + \eta^{\alpha\delta}\mathcal{J}^{\beta\gamma}$$
  3. Deformed composition law
$$(p\oplus q)^\kappa_\mu \;=\; p_\mu\left(\sqrt{1+\frac{q^2}{\Lambda^2}}+\frac{q_0}{\Lambda}\right)+q_\mu+n_\mu\left[\frac{\sqrt{1+p^2/\Lambda^2}-p_0/\Lambda}{1-\vec{p}^2/\Lambda^2}\left(q_0+\frac{q_\alpha\eta^{\alpha\beta}p_\beta}{\Lambda}\right)-q_0\right]$$
  4. **Noncommutativity** of the space-time coordinates $\rightarrow [x^0, x^i] = -ix^i/\Lambda$

[4] A. Borowiec and A. Pachol. *J. Phys. A 43* (2010)

2. PREVIOUS WORKS

Deformed relativistic wave equations in momentum space

- Klein–Gordon and Dirac equations already obtained in Hopf algebras[5,6]
- We are able to reproduce them from a curved momentum space[7]

$$\text{KG:} \qquad \left(f^\mu(p) g_{\mu\nu}(p) f^\nu(p) - m^2\right) \tilde{\phi}(p) = 0$$

$$\text{D:} \qquad \left(\gamma^\mu \eta_{\mu\rho} e^\rho{}_\nu(p) f^\nu(p) - m\right) \tilde{\psi}(p) = 0$$

being $\tilde{\phi}(p)$ the Fourier transform of the scalar field $\phi(x)$

$$\phi(x) = \frac{1}{(2\pi)^3} \int \mathrm{d}^4p \, e^{ix^\lambda p_\lambda} \tilde{\phi}(p) \, \delta(C(p) - m^2)$$

[5] J. Lukierski, A. Nowicki and H. Ruegg. *Phys. Lett. B 293* (1992)
[6] A. Nowicki, E. Sorace and M. Tarlini. *Phys. Lett. B 302* (1993)
[7] S.A. Franchino-Viñas and J.J. Relancio. *Class. Quant. Grav. 40* (2023)

3. OUR PROPOSAL FOR DSR QFT[8,9]

- AIM: translating the aforementioned results to **position space**
- First (incorrect) attempt for **scalar fields**:

$$S = \int \mathrm{d}^4x \frac{1}{2} \left\{ f^\mu(i\partial_x)\phi(x) g_{\mu\nu}(-i\partial_x) f^\nu(-i\partial_x)\phi(x) - m^2\phi^2(x) \right\}$$

- Problem: the metric becomes a differential operator
- **Correct approach**:

$$S = \int \frac{\mathrm{d}^4x}{2} \left\{ -\ell^\mu(-i\partial_x)\,\phi(x)\eta_{\mu\nu}\ell^\nu(-i\partial_x)\,\phi(x) - m^2\phi^2(x) \right\}$$

where $\ell^\mu(-i\partial_x) = e^\mu{}_\nu(p) f^\nu(p)\,\big|_{p\to -i\partial_x}$

[8] **J.J. Relancio, L. Santamaría-Sanz. *Phys. Rev. D* (2024) arXiv:2403.19520**
[9] **J.J. Relancio, L. Santamaría-Sanz. Accepted in *CQG* (2024) arXiv:2403.18772**

3. OUR PROPOSAL FOR DSR QFT

- $\eta_{\mu\nu}$ is at the base of the theory $\rightarrow$ **linear Lorentz invariance** $\rightarrow$ Restriction on bases:

$$\left.\begin{array}{c} C = C(p^2) \\ f^\mu(p) = \frac{1}{2}\frac{\partial C(p)}{\partial p_\mu} = p^\mu \frac{\partial C}{\partial p^2} \end{array}\right\} \Longrightarrow \boxed{f^\mu(p) = -f^\mu(-p)}$$

- Variational principle to Klein-Gordon action yields

$$\left(\ell^\mu(i\partial_x)\ell_\mu(-i\partial_x) + m^2\right)\phi(x) = 0$$

1. $C(p) = m^2$ holds **iff** $\ell^\mu(-i\partial_x) = -\ell^\mu(i\partial_x) \Longrightarrow \boxed{e^\mu{}_\nu(-p) = e^\mu{}_\nu(p)}$
2. But

$$e^\mu{}_\nu(p) := \left.\frac{\partial\left(p \oplus q\right)_\nu}{\partial q_\mu}\right|_{q\to 0} \quad \rightarrow \quad \text{only Snyder kinematics are allowed}$$

3. OUR PROPOSAL FOR DSR QFT

- We can replace $\boxed{\ell^\mu(-i\partial_x) \to -i\partial^\mu\Omega(-\partial^\nu\partial_\nu) \equiv -i\tilde{\partial}^\mu}$
- The **action** for KG fields is given by

$$S = \int \mathrm{d}^4x \, \frac{1}{2} \left\{ \tilde{\partial}^\mu\phi(x)\tilde{\partial}_\mu\phi(x) - m^2\phi^2(x) \right\} \qquad \text{for}$$

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2p_0}} \left(a_{\mathbf{p}} e^{-ix^\lambda p_\lambda} + a^\dagger_{\mathbf{p}} e^{ix^\lambda p_\lambda} \right)$$

- **Equations of motion**

$$\left(\tilde{\partial}^\mu\tilde{\partial}_\mu - m^2 \right)\phi(x) = 0$$

- The **energy-momentum tensor** is

$$T_{\mu\nu} = \tilde{\partial}_\mu\phi\tilde{\partial}_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\left(\tilde{\partial}^\rho\phi\tilde{\partial}_\rho\phi - m^2\phi^2 \right) \quad \text{with} \quad \partial^\mu T_{\mu\nu} = 0$$

3. OUR PROPOSAL FOR DSR QFT

Important remarks

- From DSR one finds an infinite derivative (nonlocal) QFT
- Nonlocal phenomena are already present in DSR: noncommutative spacetime and relative locality
- QFT of causal set and string theories are embedded in our scheme:

$$
\begin{aligned}
C^{CT}(-i\partial_x) &= -\Box + \frac{3}{2\pi\sqrt{6}}\frac{\Box^2}{\Lambda^2}\left[3\gamma - 2 + \ln\left(\frac{3\Box^2}{2\pi\Lambda^4}\right)\right] + \cdots \\
C^{ST}(-i\partial_x) &= -\Box\, e^{\Box/\Lambda^2}
\end{aligned}
$$

3. OUR PROPOSAL FOR DSR QFT

- The **action for Dirac fields** is given by

$$S = \int \mathrm{d}^4x\, \bar{\psi}(x) \left(i\gamma^\mu \,\tilde{\partial}_\mu - m \right) \psi(x)$$

- Variation principle leads to **equations of motion**

$$\left(i\gamma^\mu \tilde{\partial}_\mu - m \right) \psi(x) = 0$$

- The **energy-momentum tensor** is

$$T_{\mu\nu} = i\bar{\psi}(x)\gamma_\mu \tilde{\partial}_\nu \psi(x) - \eta_{\mu\nu}\bar{\psi}(x) \left(i\gamma^\rho \tilde{\partial}_\rho - m \right) \psi(x)$$

3. OUR PROPOSAL FOR DSR QFT

- The deformed **EM tensor** is

$$\tilde{F}_{\mu\nu} = \tilde{\partial}_\mu A_\nu - \tilde{\partial}_\nu A_\mu = \Omega(-\Box) F_{\mu\nu}$$

- The **action** when adding a minimal coupling to matter is

$$S_{EM} = -\int \mathrm{d}^4x \left(\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + j^\mu A_\mu \right)$$

- The deformed **Maxwell equations** are

$$\tilde{\partial}^\mu \tilde{F}_{\mu\nu} = j_\nu \,, \qquad \tilde{\partial}^\mu {}^*\tilde{F}_{\mu\nu} = \tilde{\partial}^\mu \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \tilde{F}_{\rho\sigma} = 0$$

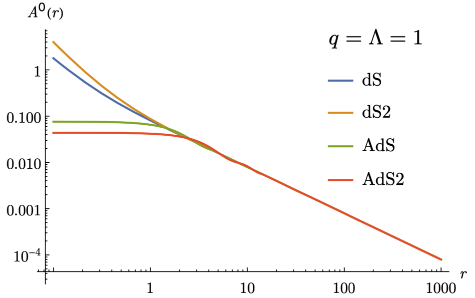
- The **electromganetic energy tensor** is given by

$$T_{\mu\nu} = \frac{1}{4} \eta_{\mu\nu} \tilde{F}_{\rho\sigma} \tilde{F}^{\rho\sigma} + \tilde{F}_{\mu\rho} \tilde{F}^{\rho}{}_{\nu} \,.$$

Electric potential of a point particle $\quad j^0(x) = q\,\delta^3(\vec{r}),\ j^i = 0$

$$A^0(r) \;=\; -\frac{q}{2\pi^2}\int_0^\infty \mathrm{d}k\,\frac{k^2}{C(-\vec{k}^2)}\,\frac{\sin(kr)}{kr}\,, \qquad \vec{E} = -\vec{\nabla}A^0(r)$$

For some metrics the electric potential becomes finite at the origin!



for the Casimirs of the AdS metrics

$$g_{\mu\nu}^{\mathrm{AdS}} = \eta_{\mu\nu}\left(1 \pm \frac{p^2}{4\Lambda^2}\right)^2, \qquad g_{\mu\nu}^{\mathrm{AdS2}} = \eta_{\mu\nu} \pm \frac{p_\mu p_\nu}{\Lambda^2}$$

4. CONCLUSIONS AND FURTHER WORK

- We have made a new proposal of **QFT in DSR** theories based on **geometry of a curved momentum space**
- This proposal leads to an infinite derivative (**nonlocal**) QFT
- Only **Snyder kinematics** are allowed in this scheme
- The electric potential for a point particle and the magnetic potential of a magnetic dipole becomes finite at the origin for **AdS models**
- **Future work**: include interactions, gravitational field of a point-like mass

THANK YOU FOR YOUR ATTENTION