Zebras Run Fast: Zernike-based Radon Transforms in Direct Detection of Dark Matter

Sebastian Sassi*, Aula Al-Adulrazzaq*, Matti Heikinheimo*, Kimmo Tuominen*

*University of Helsinki

sebastian.k.sassi@helsinki.fi

UNIVERSITY OF HELSINKI

Motivation

The event rate for a general dark matter (DM) scattering process with momentum transfer $\boldsymbol{q}$ and energy transfer E_q can be written as [9]

$$\frac{dR}{dE} \sim \int T(E, E_q, \boldsymbol{q})\mathcal{R}[f](s + \boldsymbol{v}_{\text{lab}} \cdot \hat{\boldsymbol{q}}, \hat{\boldsymbol{q}})\, d^3q\, dE_q, \tag{1}$$

where $T(E, E_q, \boldsymbol{q})$ represents a generalized target response, and $\mathcal{R}[f]$ is the Radon transform of the velocity distribution,

$$\mathcal{R}[f](t, \hat{\boldsymbol{q}}) = \int \delta(t - \boldsymbol{v} \cdot \hat{\boldsymbol{q}}) f(\boldsymbol{v})\, d^3v,$$

with $s = q/2m_\chi + E_q/q$. The high-dimensional integral in equation (1) poses a challenge for study of DM scattering in anisotropic solid-state detectors, and for non-SHM velocity distributions. Conventional numerical integration methods are inefficient and impede studies such as comprehensive parameter space scans, full-year daily modulation analyses, and studies of impact of detector orientation on the modulation. More efficient methods of integrating the event rate are therefore needed.

Note

Under common circumstances in electron scattering, conventional nuclear scattering, and the Migdal effect (under soft and free-ion approximation), the energy transfer integral becomes trivial

$$\begin{aligned} T_{\text{el}}(E, E_q, \boldsymbol{q}) &= \delta(E - E_q) T_{\text{el}}(E, \boldsymbol{q}), \\ T_{\text{nuc}}(E, E_q, \boldsymbol{q}) &= \delta(E - E_q) T_{\text{nuc}}(E, \boldsymbol{q}), \\ T_{\text{mig}}(E, E_q, \boldsymbol{q}) &= \delta\left(E_q - E - \frac{q^2}{2m_\chi}\right) T_{\text{mig}}(E, \boldsymbol{q}) \end{aligned}$$

Method

We propose a novel algorithm for evaluating the event rate integral (1) based on an expansion of the DM velocity distribution $f(\boldsymbol{v})$ in a basis of 3D Zernike functions [8]

$$f(\boldsymbol{v}) = \sum_{nlm} f_{nlm} Z_{nlm}(\boldsymbol{v}).$$

The Radon transform of the Zernike functions have a known closed form expression [3, 6], which allows writing down a closed form solution for the Radon transform of the velocity distribution,

$$\mathcal{R}[f](s + \boldsymbol{v}_{\text{lab}} \cdot \hat{\boldsymbol{q}}, \hat{\boldsymbol{q}}) = \sum_{nlm} \hat{f}_{nlm} P_n(s + \boldsymbol{v}_{\text{lab}} \cdot \hat{\boldsymbol{q}}) Y_{lm}(\hat{\boldsymbol{q}}),$$

with the Radon transform coefficients $\hat{f}_{nlm}$ given by a linear combination of the Zernike expansion coefficients

$$\hat{f}_{nlm} = \left(\frac{f_{n,lm}}{2n+3} - \frac{f_{n-2,lm}}{2n-1}\right).$$

For a suitable choice of integration coordinates, this allows solving the angle-integrated, weighted Radon transform

$$\begin{aligned} \overline{\mathcal{R}}_T[f](E, E_q, q, \boldsymbol{v}_{\text{lab}}) &\equiv \int T(E, E_q, \boldsymbol{q}) \mathcal{R}[f](s + \boldsymbol{v}_{\text{lab}} \cdot \hat{\boldsymbol{q}}, \hat{\boldsymbol{q}})\, d\Omega_q \\ &= \sum_{nl} \hat{f}^T_{nl0}(E, E_q, q) A_{nl}(v_{\text{lab}}, s), \end{aligned}$$

with

$$A_{nl}(v_{\text{lab}}, s) = 2\pi N_l \int P_n(s + v_{\text{lab}} z) P_l(z)\, dz,$$

such that the event rate can be written as

$$\frac{dR}{dE} \sim \int \overline{\mathcal{R}}_T[f](E, E_q, q, \boldsymbol{v}_{\text{lab}}) q^2\, dq\, dE_q.$$

Note

To multiply together the target response and the Radon transform coefficients, they are rotated to a common coordinate system with z-axis in the direction of $\boldsymbol{v}_{\text{lab}}$ using spherical harmonic transforms.

$$T(\cdots, \hat{\boldsymbol{q}}) \xrightarrow[O(N^3)]{\text{SHT}} T_{lm}(\cdots) \xrightarrow[O(N^3)]{\text{rotation}} T'_{lm}(\cdots) \xrightarrow[O(N^3)]{\text{SHT}^{-1}} T'(\cdots, \hat{\boldsymbol{q}})$$

$$\downarrow$$

$$\hat{f}_{nlm} \xrightarrow[O(N^4)]{\text{rotation}} \hat{f}'_{nlm} \xrightarrow[O(N^4)]{\text{SHT}^{-1}} \hat{f}'_n(\hat{q}) \xrightarrow[O(N^3)]{\times T} \hat{f}^T_n(\cdots, \hat{\boldsymbol{q}}) \xrightarrow[O(N^3)]{\text{SHT}^*} \hat{f}^T_{nl0}(\cdots)$$

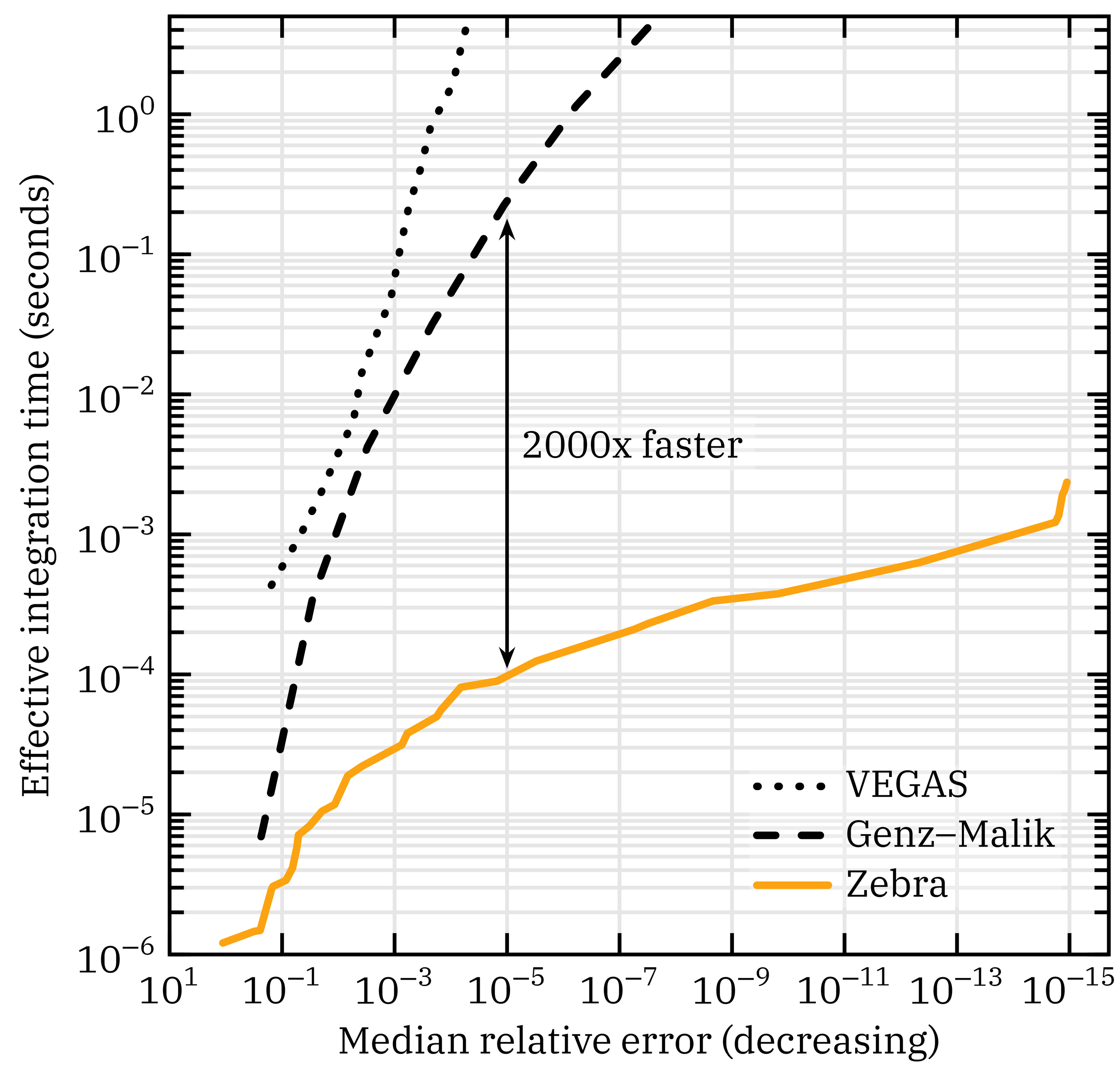

Figure 1: Time to evaluate the angle-integrated, weighted Radon transform integral of the Zernike-based method compared to two conventional numerical integration algorithms: VEGAS Monte Carlo [4, 5] and adaptive Genz–Malik quadrature [2].

Zernike Functions

The 3D Zernike functions $Z_{nlm}(\boldsymbol{v}) = R_{nl}(v) Y_{lm}(\hat{v})$ are a family of functions orthogonal on the unit sphere [1, 3, 7],

$$\int_{|\boldsymbol{v}| \le 1} Z_{nlm}(\boldsymbol{v}) Z_{n'l'm'}(\boldsymbol{v})\, d^3v = N_{nlm} \delta_{nn'} \delta_{ll'} \delta_{mm'},$$

consisting of a radial polynomial R_{nl} and a spherical harmonic Y_{lm}. Fast computation of the expansion coefficients

$$f_{nlm} = \int f(\boldsymbol{v}) Z_{nlm}(\boldsymbol{v})\, d^3v$$

is enabled by extension of standard methods from literature on spherical harmonic transforms. The function $f(\boldsymbol{v})$ is sampled on a grid with appropriately spaced nodes. The 3D integral is then performed via a fast Fourier transform (FFT) followed by two Gauss–Legendre quadrature (GLQ) steps

$$f(v_i, \theta_j, \varphi_k) \xrightarrow[O(N^3 \log N)]{\text{FFT}} f_m(v_i, \theta_j) \xrightarrow[O(N^4)]{\text{GLQ}} f_{lm}(v_i) \xrightarrow[O(N^4)]{\text{GLQ}} f_{nlm}.$$

Library Implementation

We have implemented the Zernike-based Radon transform method in the library **`ZebraDM`**. This library aims to become a general-purpose, high-performance tool for theoretical analysis of direct detection.

`https://github.com/sebsassi/zebradm.git`

References

[1] N. Canterakis. *Proceedings of the 11th Scandinavian Conference on Image Analysis*. 1997, pp. 85–93.
[2] A. Genz and A. Malik. *Journal of Computational and Applied Mathematics* 6.4 (1980), pp. 295–302.
[3] A. J. E. M. Janssen. *arXiv e-prints*, arXiv:1510.04837 (Oct. 2015).
[4] G. P. Lepage. preprint CLNS 80-447. Cornell, 1980.
[5] G. P. Lepage. *Journal of Computational Physics* 27.2 (May 1978), pp. 192–203.
[6] A. K. Louis. *SIAM Journal on Mathematical Analysis* 15.3 (1984), pp. 621–633.
[7] R. J. Mathar. *Baltic Astron.* 17 (2008), p. 383.
[8] S. Sassi, A. Al-Adulrazzaq, M. Heikinheimo, et al. *Phys. Rev. D* 112 (4 Aug. 2025), p. 043003.
[9] T. Trickle, Z. Zhang, K. M. Zurek, et al. *JHEP* 03 (2020), p. 036.