Unitarity in the non-relativistic regime and implications for dark matter

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Identification of Dark Matter 2026

Based on work with Kalliopi Petraki (LPENS)

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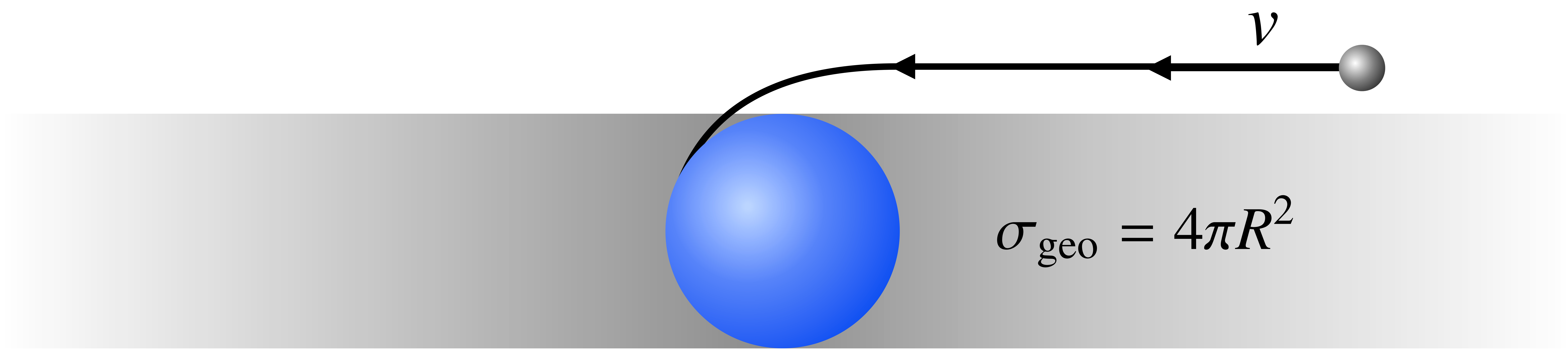
UNIVERSITY
OF OSLO

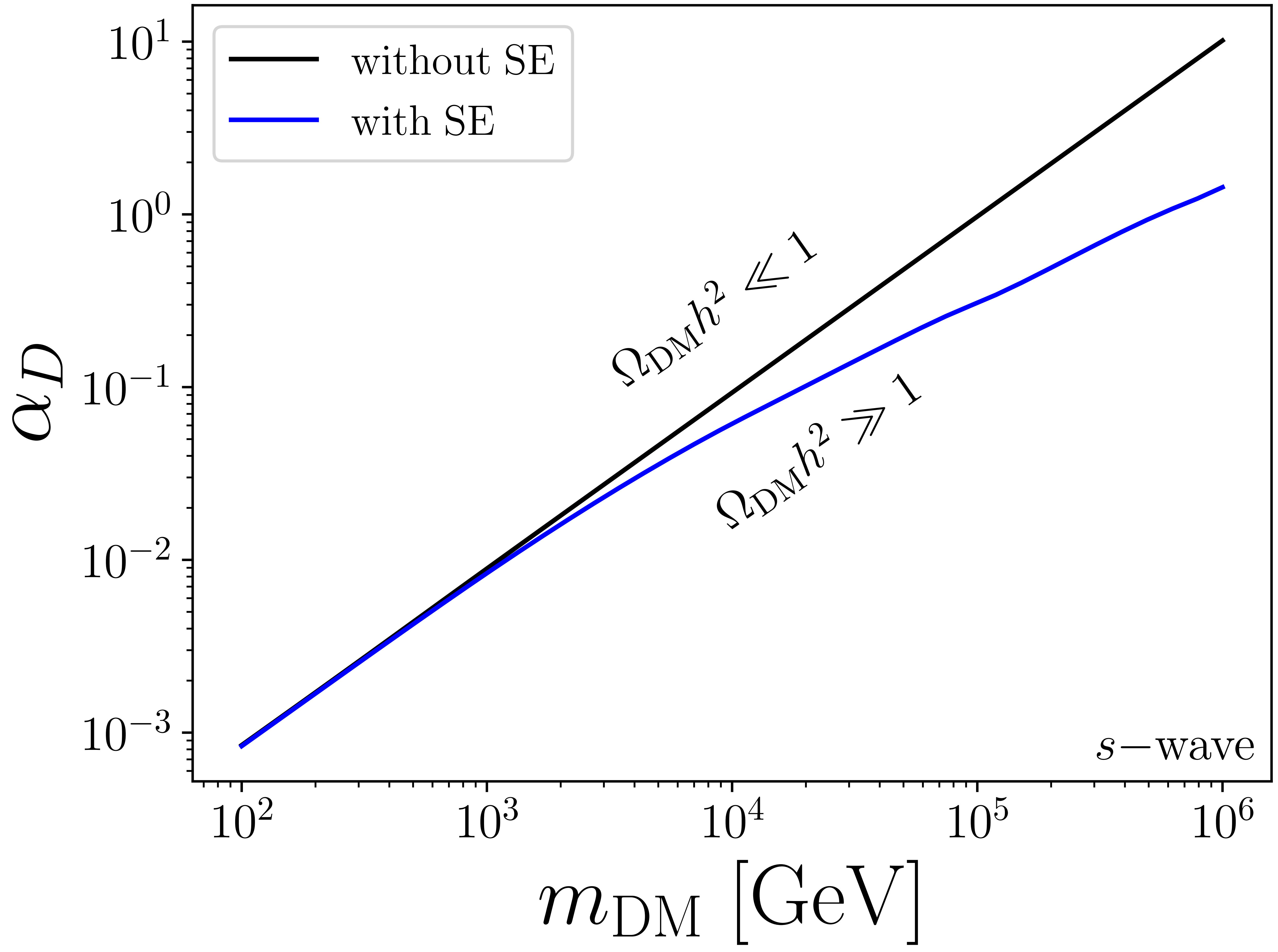
Introduction: Long-range interactions



$$\sigma_{\text{eff}} = \sigma_{\text{geo}} \times \left[1 + \left(\frac{v_{\text{esc}}}{v}\right)^2\right]$$

[Arkani-Hamed, et. al.; 0810.0713]

Long-range interactions & dark matter



$$\sigma_{\rm ann}^{\ell=0} v_{\rm rel} \simeq \frac{\pi\alpha_D^2}{M_{\rm DM}^2} \times \underbrace{\frac{2\pi\alpha_D/v_{\rm rel}}{1-\exp(-2\pi\alpha_D/v_{\rm rel})}}_{\text{Sommerfeld effect}}$$

Crucially:

$$\Omega_{\rm DM}h^2 \propto 1/\langle\sigma_{\rm ann}v_{\rm rel}\rangle = 0.120 \pm 0.001$$

$\Longrightarrow$ Implications for unitarity bounds

[Arkani-Hamed, et. al.; 0810.0713]
[Petraki, Postma, Wiechers; 1505.00109]

Unitarity review

"probabilities sum to 1"

$$SS^\dagger = S^\dagger S = \mathbb{1}$$

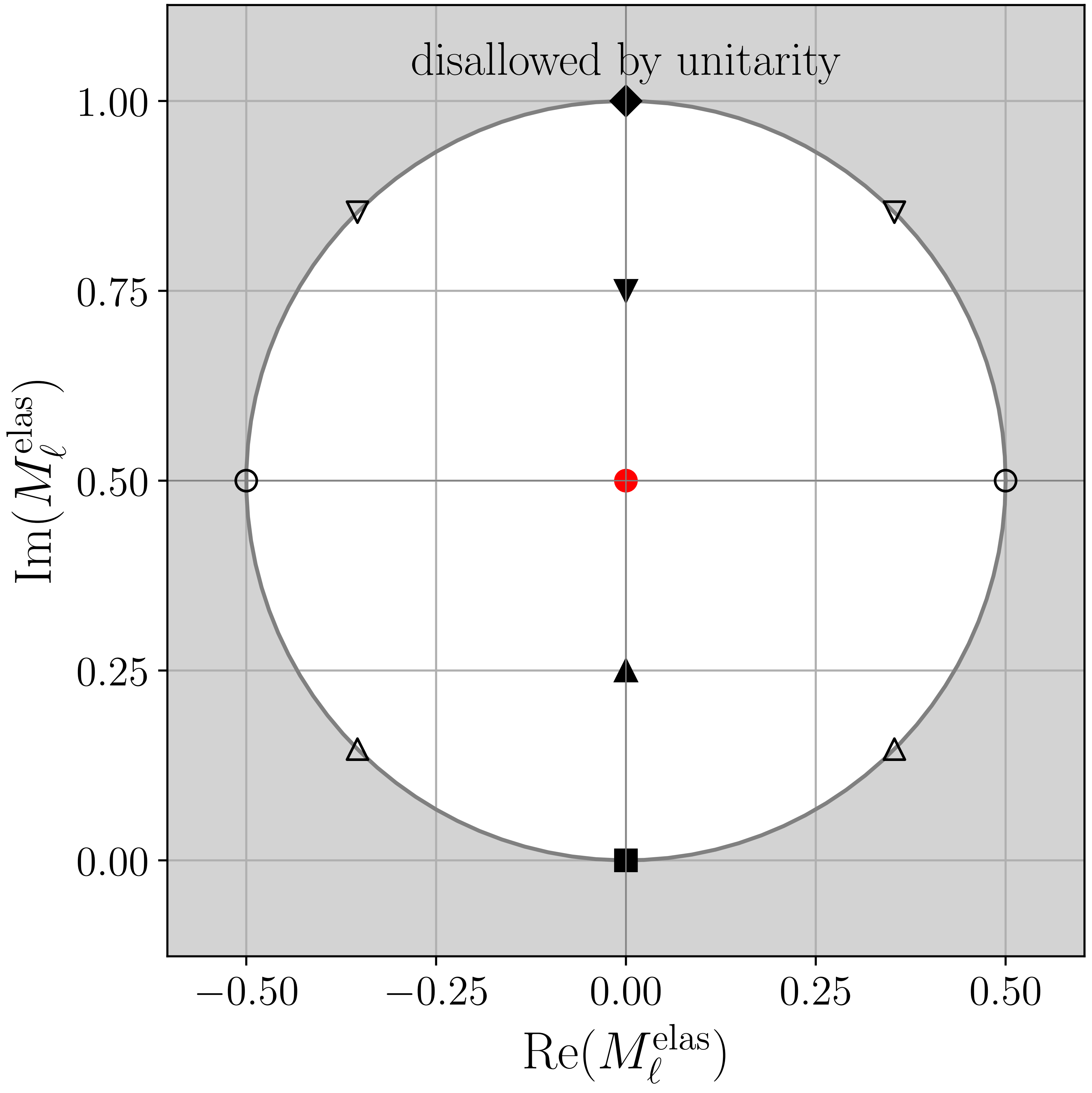
$$\Downarrow$$

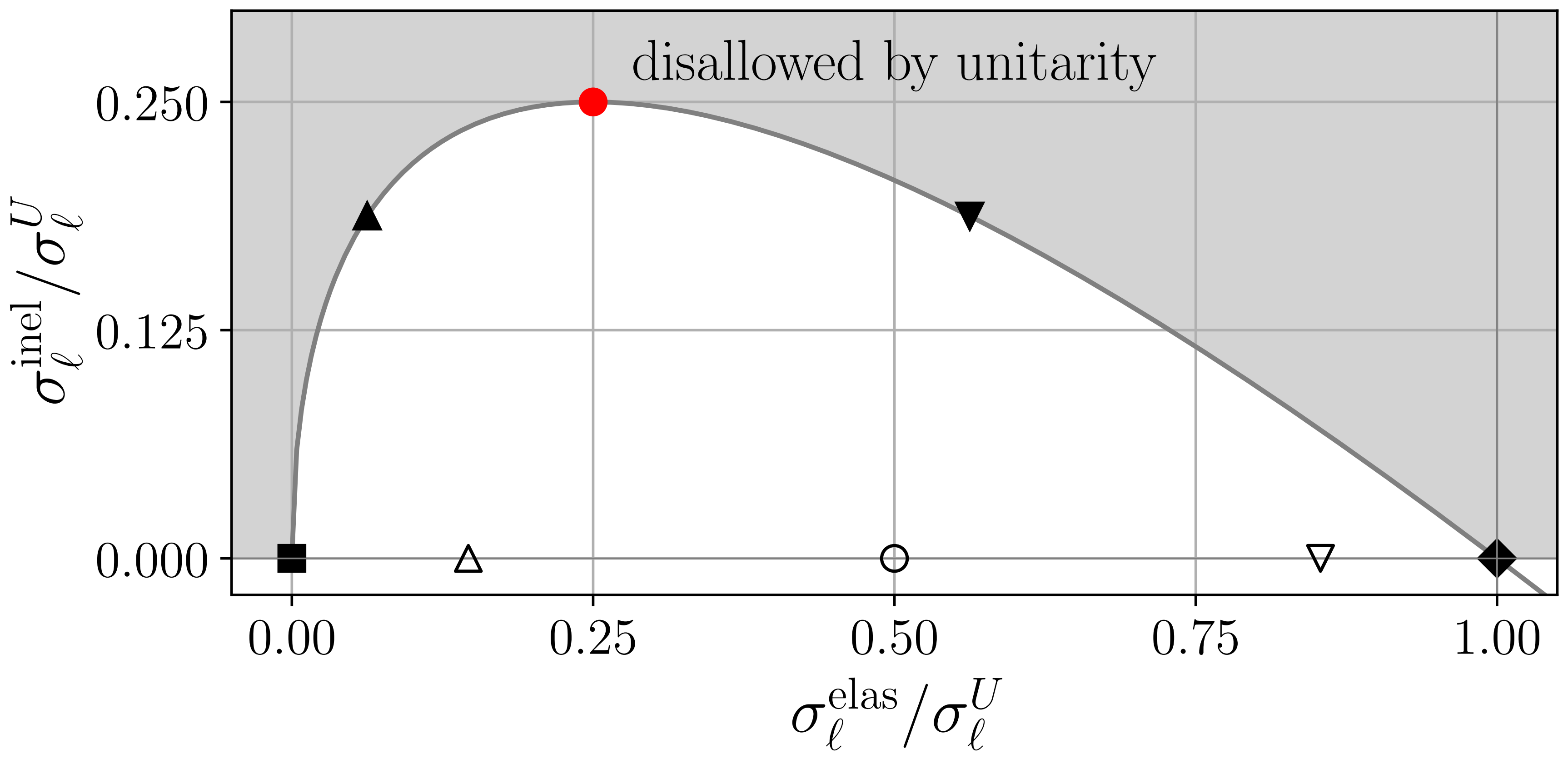
$$\sigma_\ell^{\text{inel}}/\sigma_\ell^U \leq \sqrt{\sigma_\ell^{\text{elas}}/\sigma_\ell^U}\left(1 - \sqrt{\sigma_\ell^{\text{elas}}/\sigma_\ell^U}\right)$$

$$\Downarrow$$

$$\sigma_\ell^{\text{inel}}/\sigma_\ell^U \leq \frac{1}{4} \quad \& \quad \sigma_\ell^{\text{elas}}/\sigma_\ell^U \leq 1$$

$$\sigma_\ell^U = \frac{c_\ell 4\pi(2\ell+1)}{\mathbf{k}^2}$$ "unitarity limit"





Long-range interactions & unitarity

- ***Violation of partial wave unitarity*** is intimately related to the emergence of long-range physics.

- Unitarity puts a limit on partial-wave cross sections:

$$\sigma^{\ell}_{\rm inel} v_{\rm rel} \leq \sigma^{\ell}_{\rm uni} v_{\rm rel} = \frac{4\pi(2\ell+1)}{M^2_{\rm DM} v_{\rm rel}}$$

- Historically, unitarity has been used to place limits on the DM mass

$$\sigma^{\ell=0}_{\rm can} v_{\rm rel} = 2.2 \times 10^{-26}\ {\rm cm}^3/{\rm s} \leq \frac{4\pi}{M^2_{\rm DM} v_{\rm rel}}$$

?

[Griest, Kamionkowski (1990)]

Long-range interactions & unitarity

- The mismatch in the hints for the need for new physics

$$\sigma^{\ell}_{\rm inel} v_{\rm rel} \leq \sigma^{\ell}_{\rm uni} v_{\rm rel} = \frac{4\pi(2\ell+1)}{M_{\rm DM}^2 v_{\rm rel}}$$

🍎 ↔ 🍏

- Including Sommerfeld effect:

$$\sigma^{\ell=0}_{\rm ann} v_{\rm rel} \simeq \frac{\pi\alpha_D^2}{M_{\rm DM}^2} \times \frac{2\pi\alpha_D/v_{\rm rel}}{1-\exp(-2\pi\alpha_D/v_{\rm rel})} \xrightarrow{\alpha_D \gg v_{\rm rel}} \frac{2\pi^3\alpha_D^3}{M_{\rm DM}^2 v_{\rm rel}}$$

$$\Longrightarrow \text{ true for all partial waves}$$

$$\Longrightarrow \text{ still not the whole story}$$

[Hisano, Matsumoto, Nojiri; hep-ph/0212022]
[Baldes, Petraki; 1703.00478]

Inelastic unitary violation at low energies

- Attractive Coulomb(-like) potentials violate unitarity at ***large couplings.*** [Petraki, von Harling: 1407.7874] [Baldes, Petraki: 1703.00478]

- **Radiative bound-state formation, via emission of charged bosons**

  $\Longrightarrow$ **Unitarity violation at low couplings**

  [Oncala, Petraki: 2101.08666, 2101.08667]
  [Binder et al.: 2308.01336]
  [**MMF**, Petraki: 2602.20243]

- Potentials featuring the exchange of light, but massive mediators leading to **parametric resonances at the thresholds for bound state formation.** [e.g. Cassel: 0903.5307]

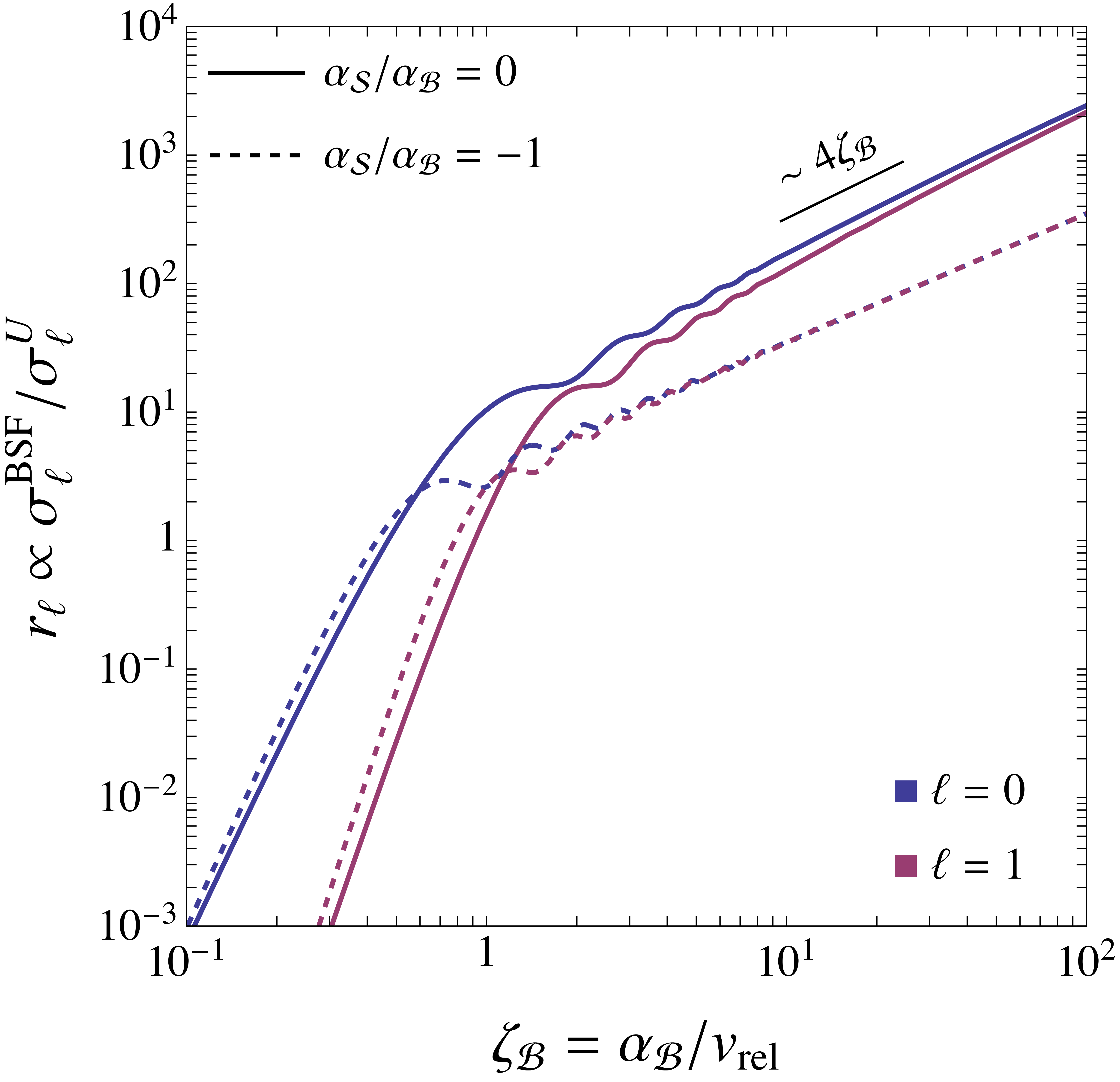
Unitarity violation in radiative bound-state formation

$$\underbrace{X + X}_{V_{\mathcal{S}}} \to \underbrace{\mathcal{B}(XX^\dagger)}_{V_{\mathcal{B}}} + \boxed{\varrho \;\; charged}$$

- Charged scalars allow for $\Delta\ell = 0$ **monopole transitions**.
- In particular, when

$$V_{\mathcal{S}} \neq V_{\mathcal{B}}$$

unitarity violation becomes possible for ***arbitrarily small couplings.***



[Oncala, Petraki: 2101.08666, 2101.08667]
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Unitarity violation in radiative bound-state formation

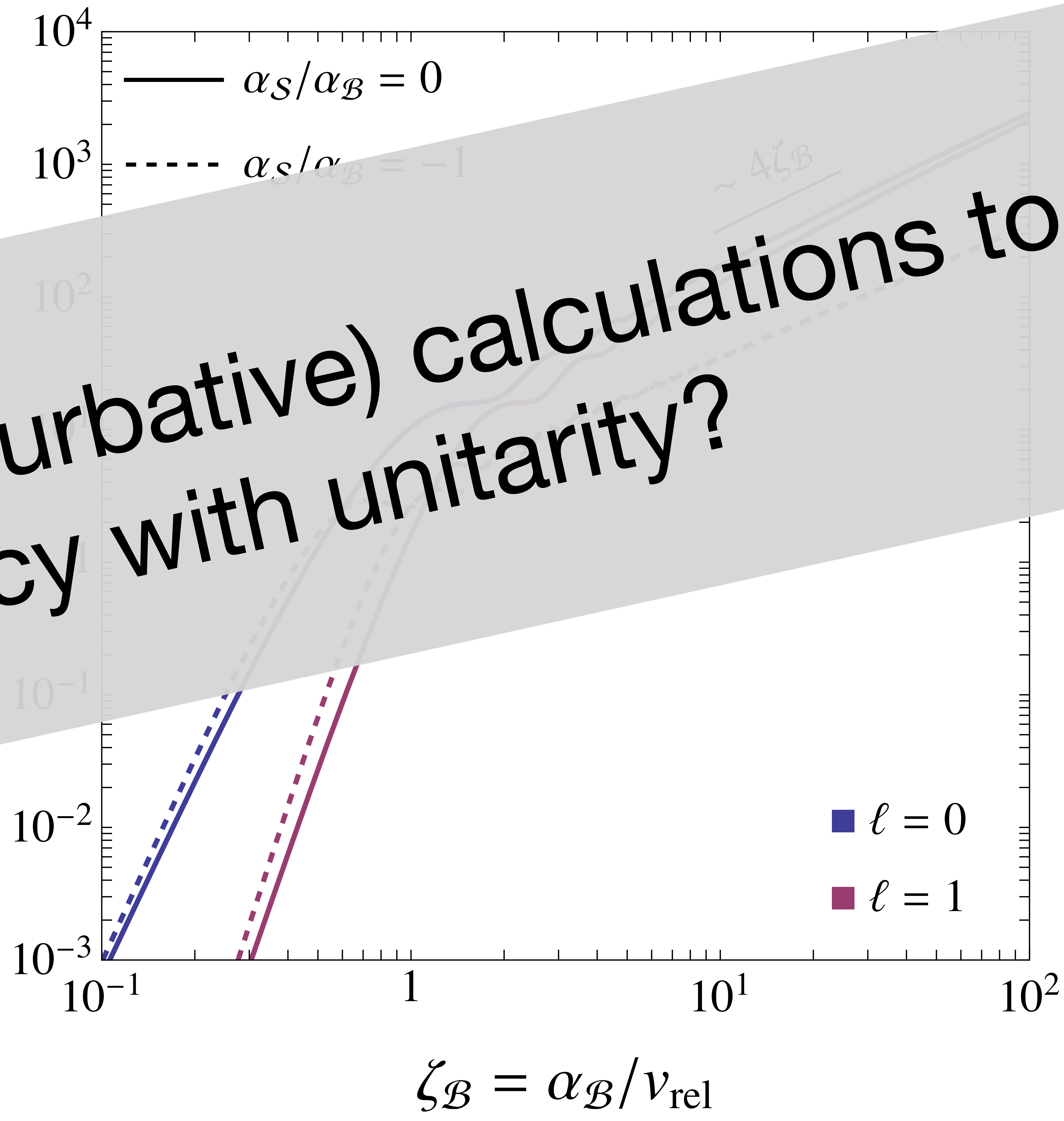
$$\underbrace{X + X}_{V_S} \rightarrow \underbrace{\mathcal{B}(XX^\dagger)}_{V_{\mathcal{B}}} + \boxed{\varrho \;\; charged}$$

- Charged scalars allow for $\Delta\ell = 0$ **monopole transitions.**
- In particular, when

$$V_S \neq V_{\mathcal{B}}$$

unitarity violation becomes possible for *arbitrarily small couplings.*

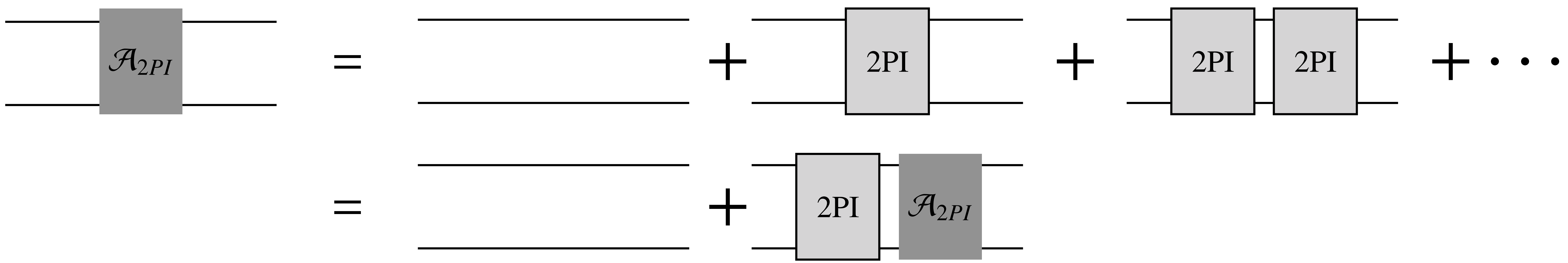
How can we improve (perturbative) calculations to **ensure** consistency with unitarity?

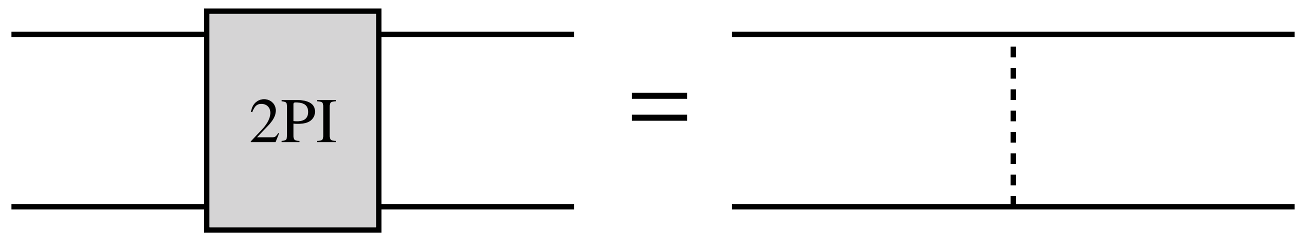


[Oncala, Petraki: 2101.08666, 2101.08667]
[Binder et al.: 2308.01336]
[**MMF**, Petraki: 2602.20243]

Resummation of long-range interactions

- In order to account for the fact that the incoming states are **no longer asymptotically free at infinity**, we now have to sum the *two-particle irreducible diagrams*



where, e.g., 

$$\Longrightarrow \quad \mathcal{M}(\mathbf{k}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \tilde{\psi}_{\mathbf{k}}(\mathbf{q})\, \mathcal{A}(\mathbf{q})$$

$\psi_{\mathbf{k}}(\mathbf{x})$: satisfies the Schrödinger equation!

[Petraki, Postma, Wiechers: 1505.00109]

Long-range interactions & resummation

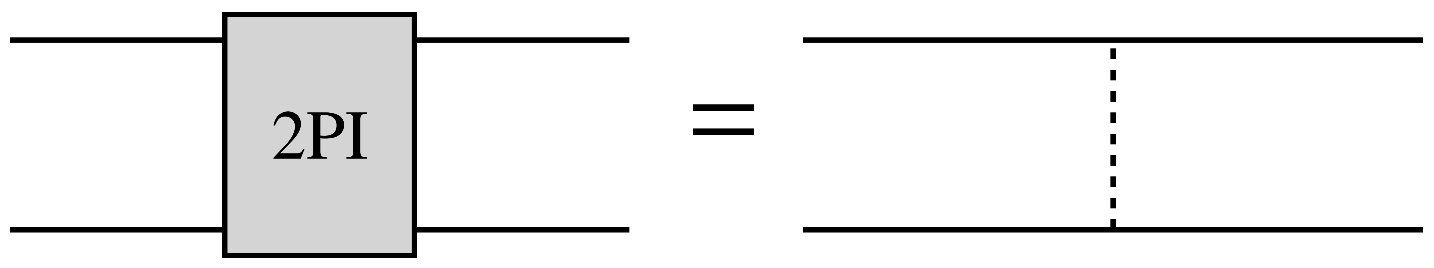
- The wave function satisfies the (momentum space) Schrödinger equation:

$$\left(\frac{\mathbf{p}^2}{2\mu} - \mathcal{E}_{\mathbf{k}}\right)\tilde{\psi}_{\mathbf{k}}(\mathbf{p}) = \frac{1}{4m_t\mu}\int\frac{d^3p'}{(2\pi)^3}\mathcal{A}^{2\mathrm{PI}}(s,\mathbf{p},\mathbf{p}')\tilde{\psi}_{\mathbf{k}}(\mathbf{p}')$$

- We define the (**non-local**) potential as

$$\mathcal{V}(\mathbf{r},\mathbf{r}') = \frac{-1}{4m_t\mu}\int\frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\,e^{i\mathbf{p}\cdot\mathbf{r}}\mathcal{A}^{2\mathrm{PI}}(s,\mathbf{p},\mathbf{p}')\,e^{-i\mathbf{p}'\cdot\mathbf{r}'}$$

- If one only considers the one-boson-exchange (OBE):



$$\Longrightarrow \quad \mathcal{V}(\mathbf{r},\mathbf{r}') \sim \frac{\alpha}{r}e^{-m_\varphi r}\delta(r-r')$$

[Petraki, Postma, Wiechers: 1505.00109]

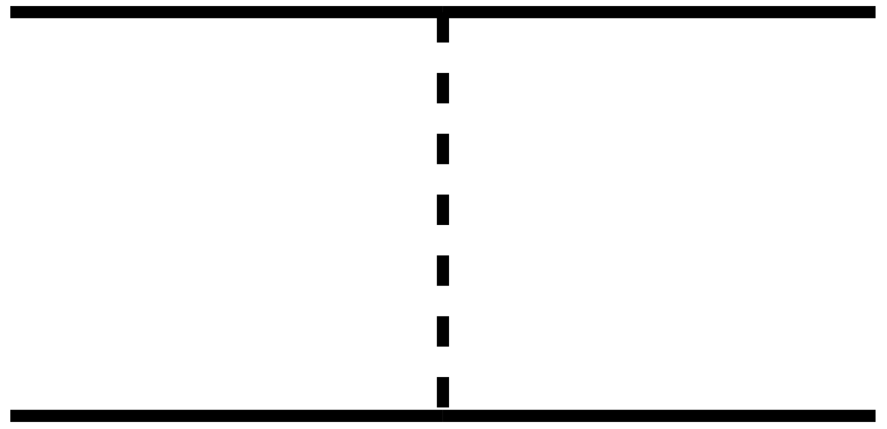
Inelastic unitary violation at low energies

Optical theorem: $$\mathrm{Im}\, M_\ell^{aa} = \sum_b \left(M_\ell^{ab*} \right) \left(M_\ell^{ab} \right)$$

$$\mathcal{O}(\alpha) \quad \text{vs.} \quad \mathcal{O}(\alpha^2)$$

Key: Resummation is required

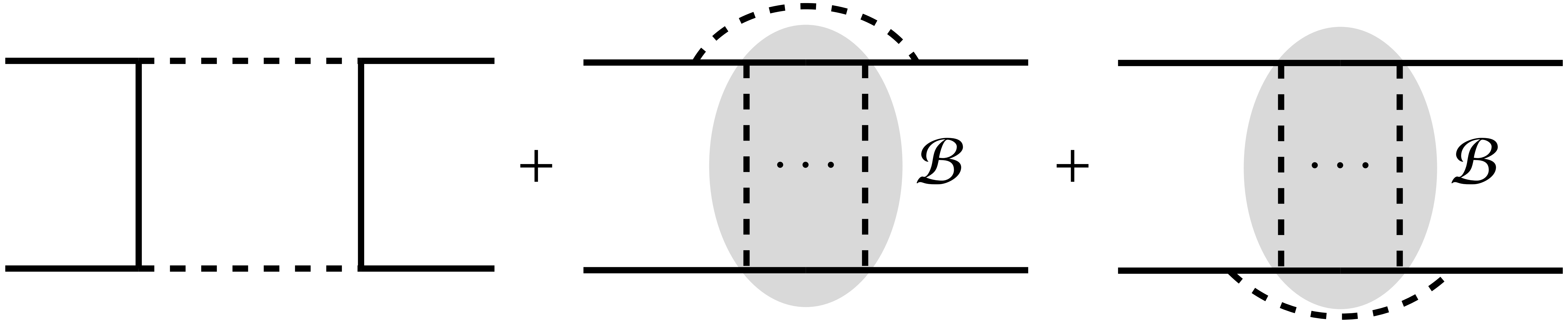
Purely elastic kernel $\Longrightarrow$ **Hermitian** potential



$\Longrightarrow$ Leads to Sommerfeld effect

$\Longrightarrow$ Unitarizes elastic cross sections

Squared inelastic kernel $\Longrightarrow$ **Anti-Hermitian** potential

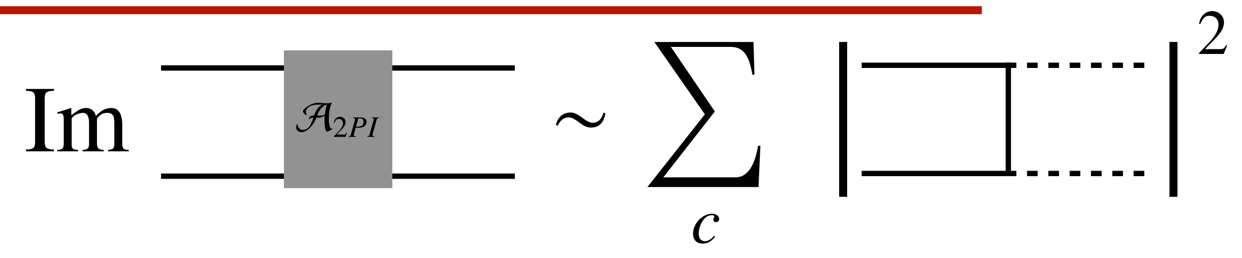


Resummation and inelastic processes

- Resummation of inelastic processes is equivalent to solving the following Schrödinger equation:

$$[\mathcal{S}_\ell(r) - \mathcal{E}_{\mathbf{k}}]u_{k,\ell}(r) = i \sum_j r v_\ell^{j*}(r) \int_0^\infty dr' \, r' \, v_\ell^j(r') u_{k,\ell}(r')$$

where

new ingredient 

$$\mathcal{S}_\ell(r) \equiv -\frac{1}{2\mu}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{2\mu\, r^2} + V(r), \qquad u_{k,\ell}(r) \equiv kr\psi_{k,\ell}(r)$$

- We solve the above equations subject to the boundary conditions:

$$\psi_{\mathbf{k}}(\mathbf{r}) \overset{r\to\infty}{\simeq} e^{i\mathbf{k}\cdot\mathbf{r}} + f_{\mathbf{k}}(\Omega_{\mathbf{r}})\frac{e^{ikr}}{r}$$

- The equation above can actually be ***solved analytically.***

Regulated & Unregulated Amplitude Relation

- Using our solution to Schrödinger's Equation with non-Hermitian potential:

$$\boxed{M^{\mathrm{inel},i}_{\ell,\mathrm{reg}}(k) = \sum_j [\mathbb{N}_\ell^{-1}(k)]^{ij}\; M^{\mathrm{inel},j}_{\ell,\mathrm{unreg}}(k).}$$

with

$$\mathbb{N}_\ell(k) = \mathbb{1} + \; M_{\ell,\mathrm{unreg}}(k)\, M^{\dagger}_{\ell,\mathrm{unreg}}(k) + i\,\mathbb{W}_\ell(k)$$

And

$$\mathbb{W}^{ij}_\ell(k) = \frac{4}{c_\ell m_T^2}\sqrt{\frac{k^i(s)k^j(s)}{c^i c^j}}\left[-\frac{\mathrm{P.V.}}{\pi}\int_{-\infty}^{\infty}\mathrm{d}q\, q^2\,\frac{\mathcal{M}^i_{\ell,\mathrm{unreg}}(q)\mathcal{M}^{j*}_{\ell,\mathrm{unreg}}(q)}{q^2-k^2} + \sum_n \frac{\mathcal{M}^{\mathrm{BSD},i}_{n\ell,\mathrm{unreg}}\mathcal{M}^{\mathrm{BSD},j\,*}_{n\ell,\mathrm{unreg}}}{(\kappa^2_{n\ell}+k^2)/(2\mu)}\right]$$

$\Longrightarrow$ $\mathbb{W}_\ell(k)$ captures contributions from bound-state formation and non-analytic behavior of the amplitudes

[**MMF**, Petraki: 2512.02097]

Unitarity restoration: resummation of inelastic interactions

$$x_{\ell,\text{(un)reg}} \equiv \sigma^{\text{elas}}_{\ell,\text{(un)reg}}/\sigma^U_\ell$$

$$y^j_{\ell,\text{(un)reg}} \equiv \sigma^{\text{inel},j}_{\ell,\text{(un)reg}}/\sigma^U_\ell$$

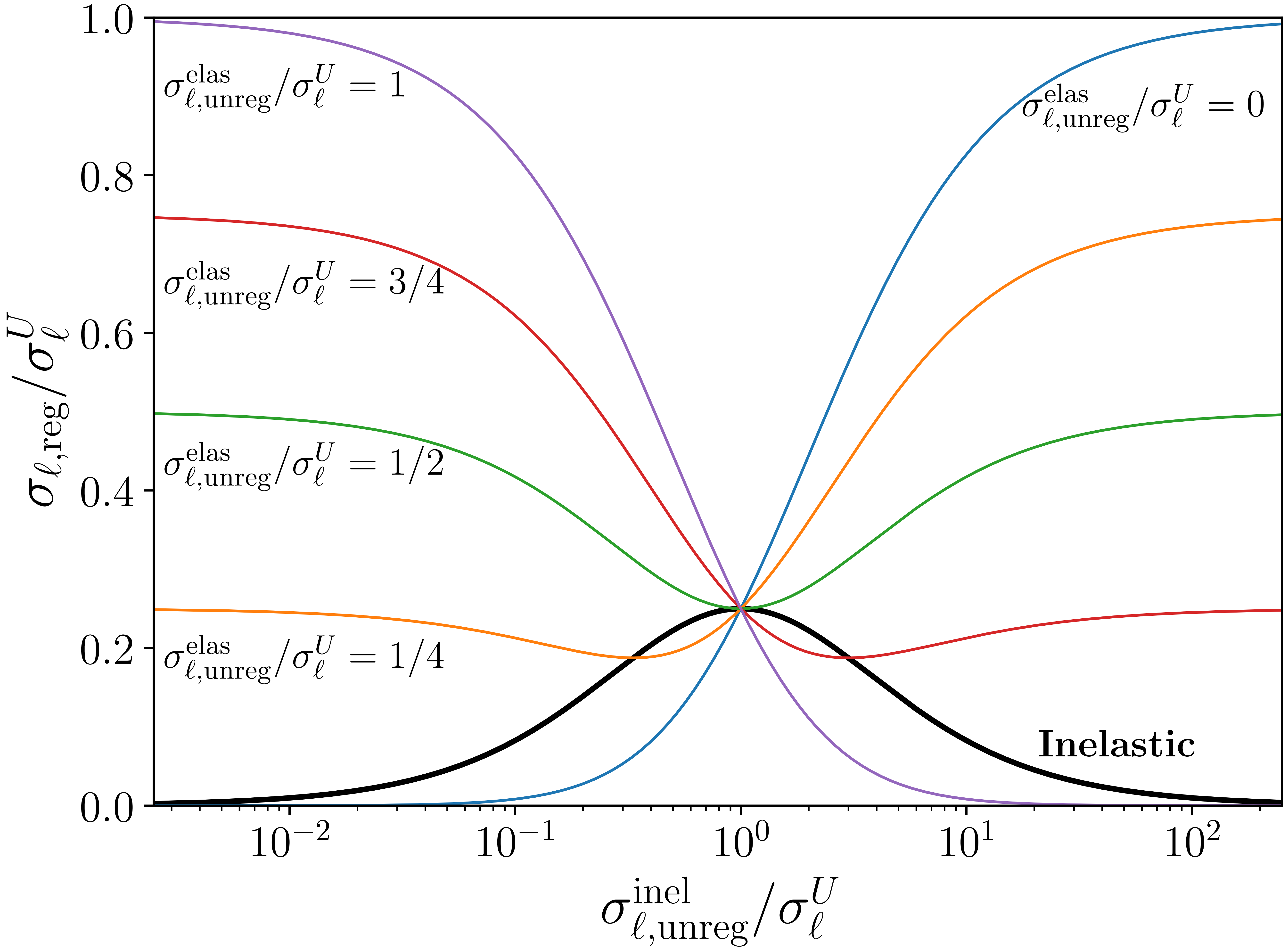
$$y_{\ell,\text{(un)reg}} \equiv \sum_j y^j_{\ell,\text{(un)reg}}$$

$$\boxed{\mathbb{W}_\ell(k) = 0}$$

$$x_{\ell,\text{reg}} = \frac{x_{\ell,\text{unreg}} + (1 - x_{\ell,\text{unreg}})y^2_{\ell,\text{unreg}}}{(1 + y_{\ell,\text{unreg}})^2}$$

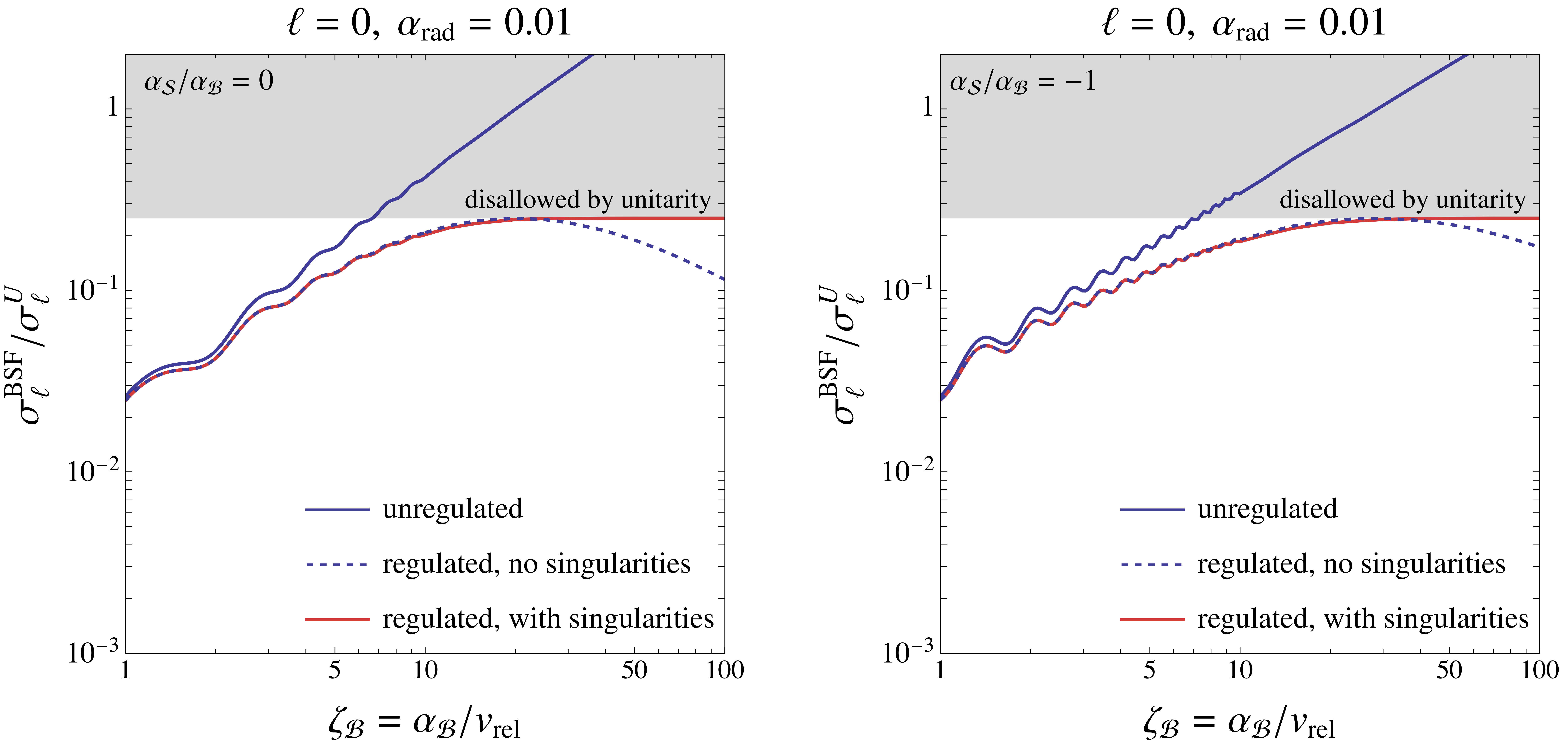
$$y^j_{\ell,\text{reg}} = \frac{y^j_{\ell,\text{unreg}}}{(1 + y_{\ell,\text{unreg}})^2}$$

[**MMF**, K. Petraki; *PLB* 858 (2024) 139022]



$\Longrightarrow$ Like the inclusion of the Sommerfeld effect, this procedure changes model predictions and **must be included** when comparing theory to observations.

Example: Radiative Bound-state formation



[**MMF**, Petraki: 2602.20243]

Unitarity restoration: resummation of inelastic interactions

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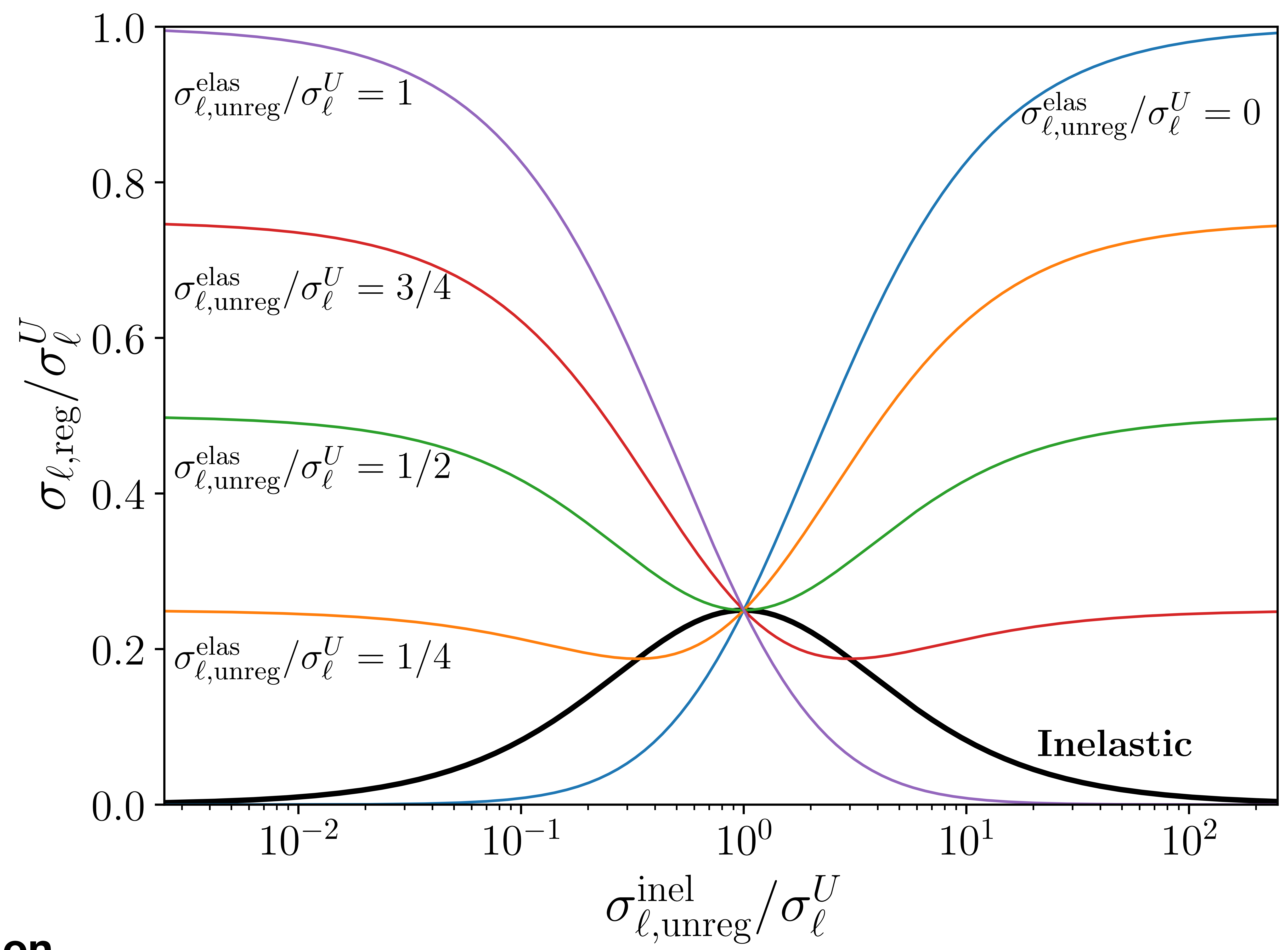

Inelastic scattering: DM production & indirect detection

- Processes that do not directly impact production or indirect signals (e.g., BSF at high temperatures) <u>**renormalize *all* inelastic channels**</u> $\Longrightarrow$ significant effect

Unitarity restoration: resummation of inelastic interactions

[**MMF**, K. Petraki; *PLB* 858 (2024) 139022]

$$x_{\ell,(\mathrm{un})\mathrm{reg}} \equiv \sigma^{\mathrm{elas}}_{\ell,(\mathrm{un})\mathrm{reg}}/\sigma^U_\ell$$

$$y^j_{\ell,(\mathrm{un})\mathrm{reg}} \equiv \sigma^{\mathrm{inel},j}_{\ell,(\mathrm{un})\mathrm{reg}}/\sigma^U_\ell$$

$$y_{\ell,(\mathrm{un})\mathrm{reg}} \equiv \sum_j y^j_{\ell,(\mathrm{un})\mathrm{reg}}$$

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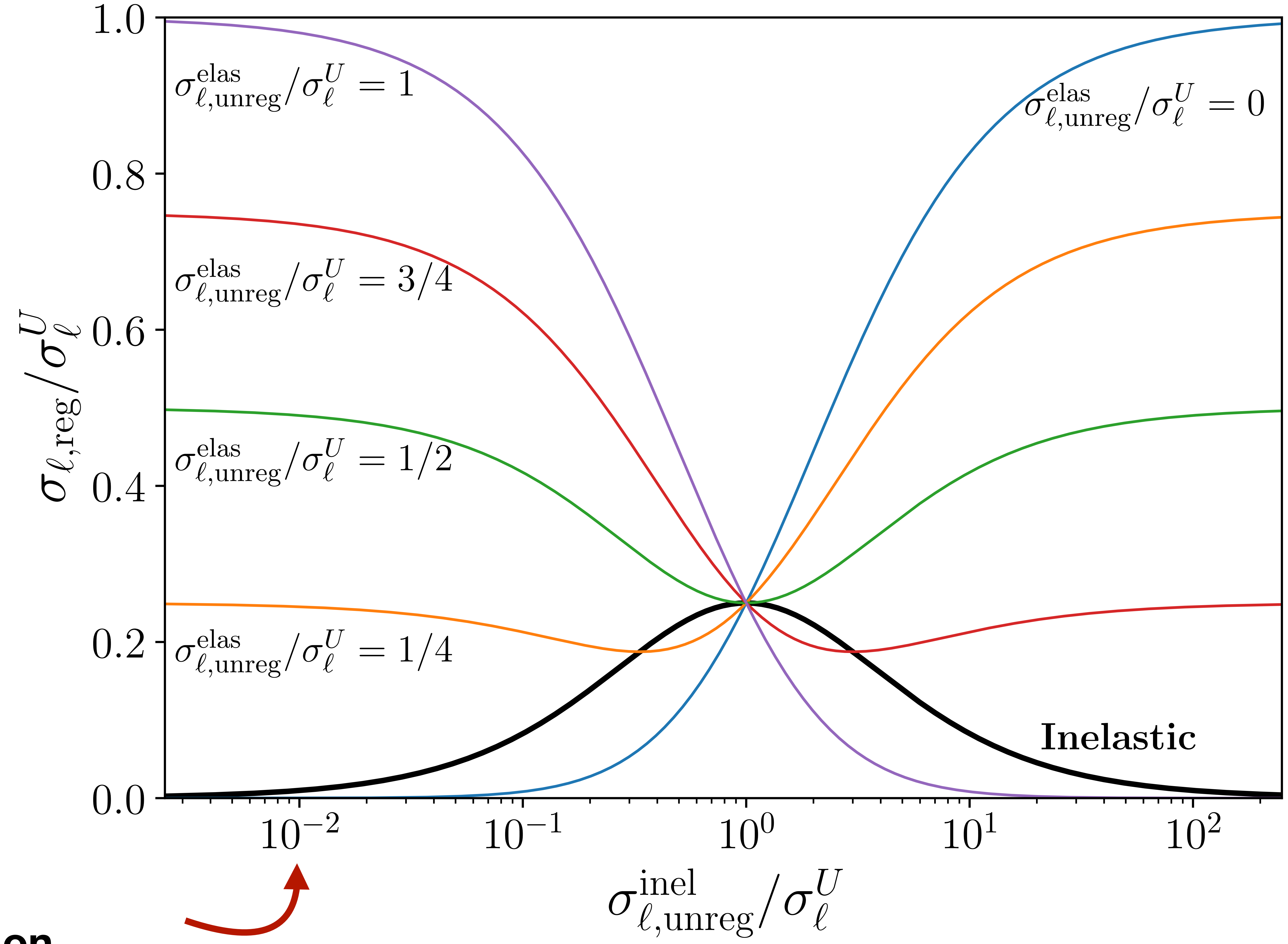

Inelastic scattering: DM production & indirect detection

- When $\sigma^{\mathrm{inel}}_{\ell,\mathrm{unreg}}/\sigma^U_\ell \sim 10^{-2}$ then the impact of the unitarization **exceeds the observational precision** of the dark matter abundance. See also [Petraki, Socha, Vasilaki: 2505.20443]

Unitarity restoration: resummation of inelastic interactions

[**MMF**, K. Petraki; *PLB* 858 (2024) 139022]

$$x_{\ell,\mathrm{(un)reg}} \equiv \sigma^{\mathrm{elas}}_{\ell,\mathrm{(un)reg}}/\sigma^U_\ell$$

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$$y^j_{\ell,\mathrm{(un)reg}} \equiv \sigma^{\mathrm{inel},j}_{\ell,\mathrm{(un)reg}}/\sigma^U_\ell$$

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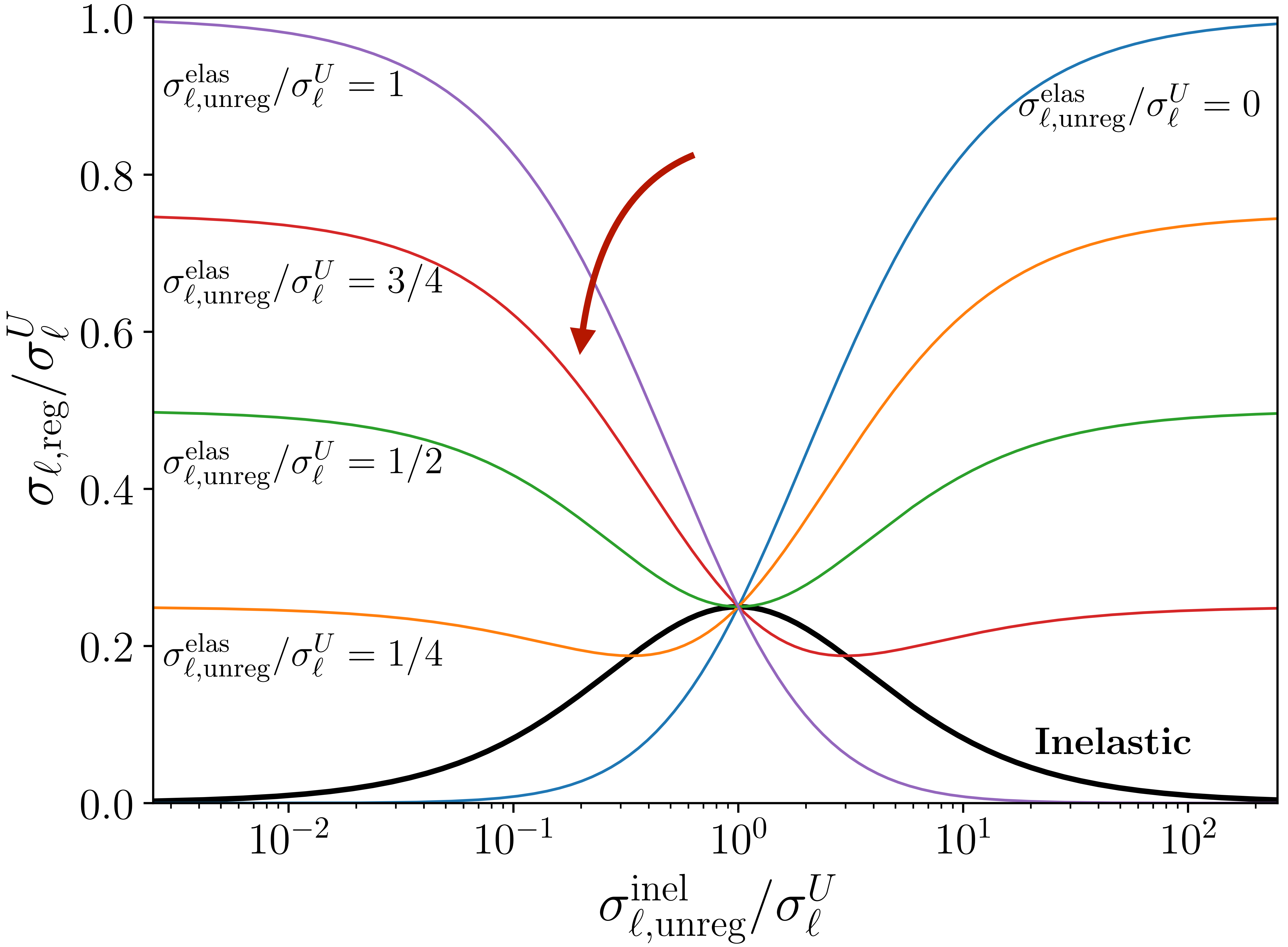


<u>Elastic scattering</u>: DM self-interactions

- <u>Inelastic scattering changes elastic scattering</u>, significantly and non-montonically.
- Changes calculations of DM self-interactions and their effect on galactic structure.

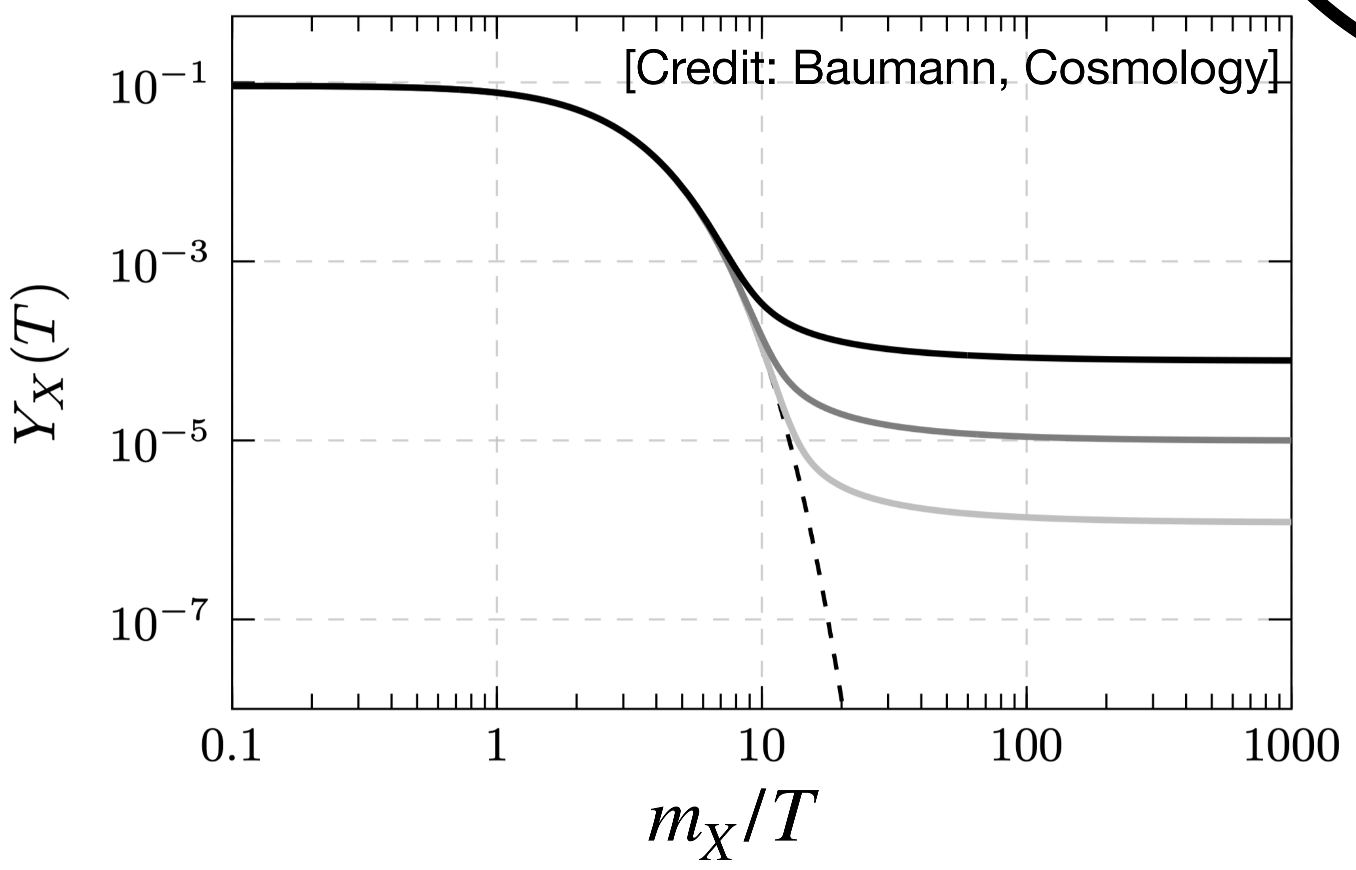
Phenomenological implications

$\sigma_\ell^{\text{elas}}$ $\sigma_\ell^{\text{inel}}$

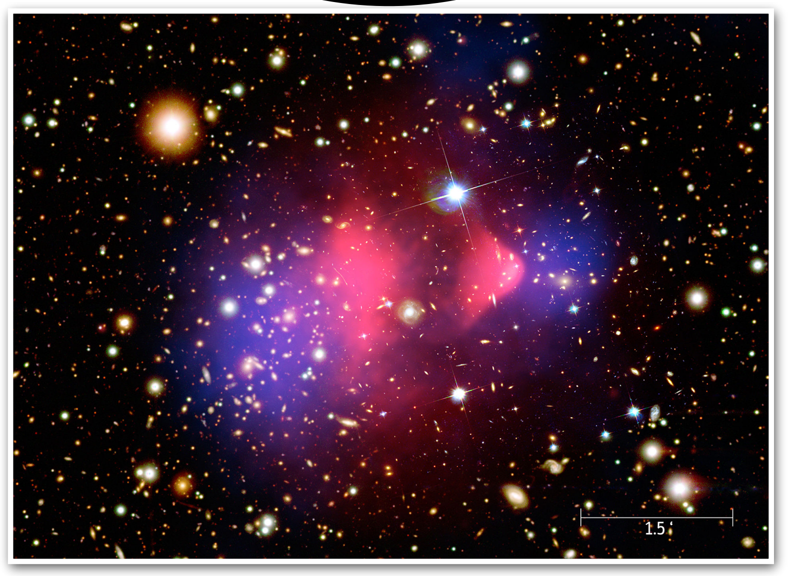
$\Omega_{\text{DM}} h^2$
DM production

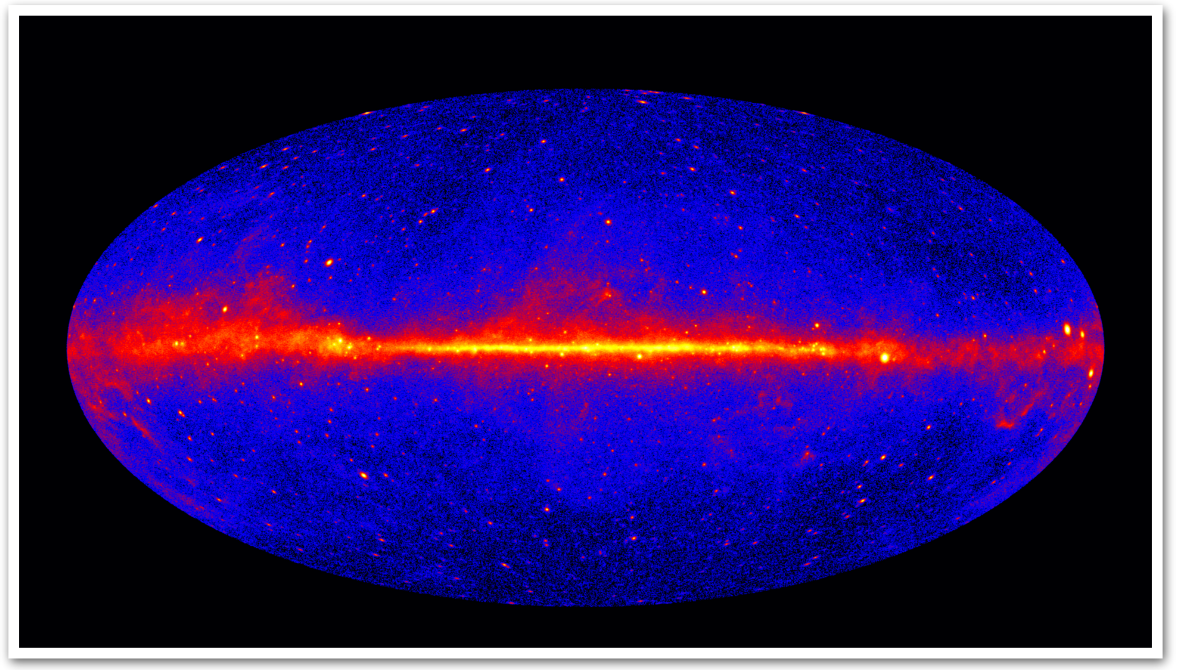
$\sigma_{\text{elas}}/M_{\text{DM}}$
Self-interacting DM

$\sigma_{\text{DM}\to\text{SM}}$
Indirect detection



[Credit: Baumann, Cosmology]

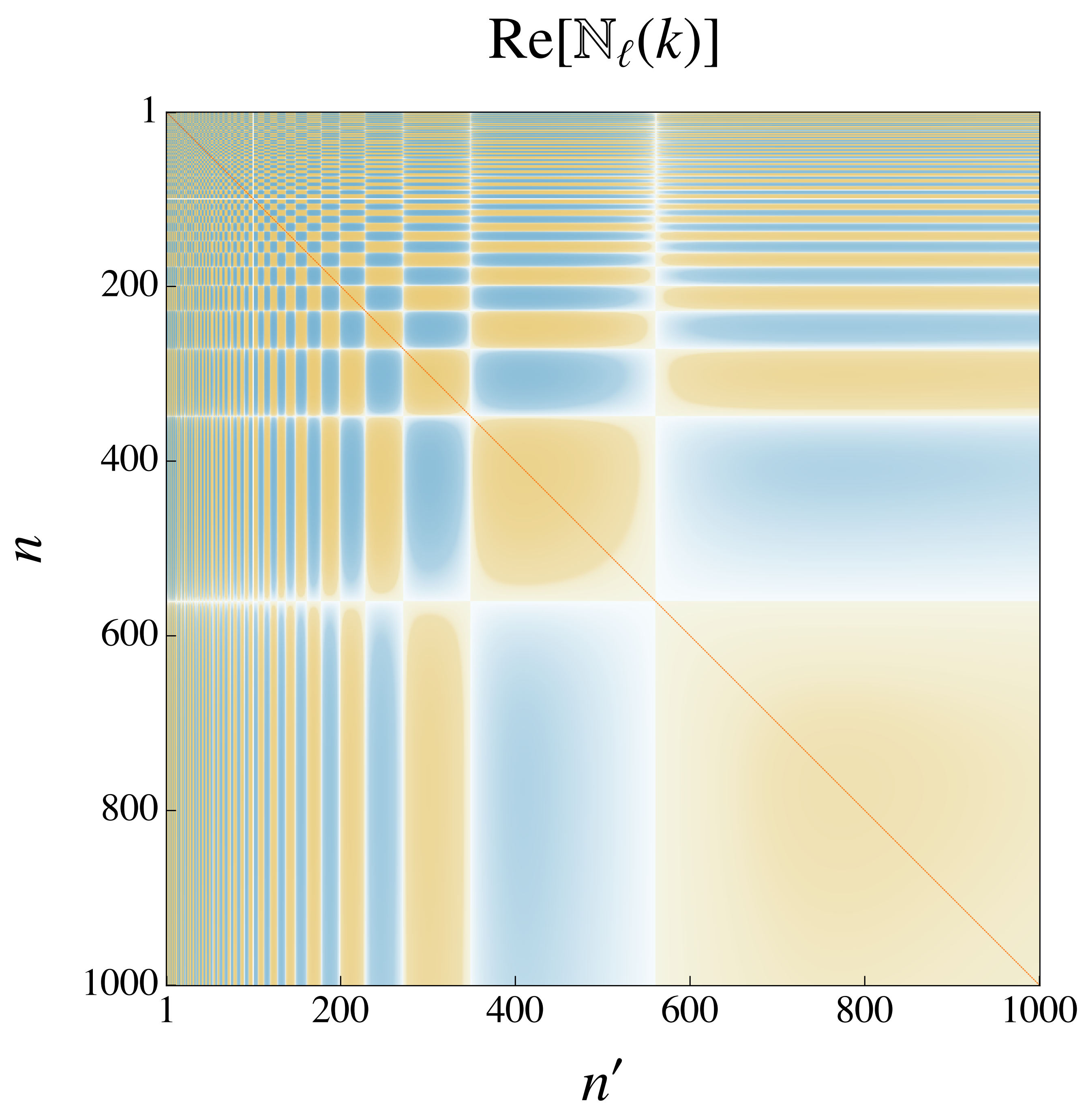

[Credit: NASA]

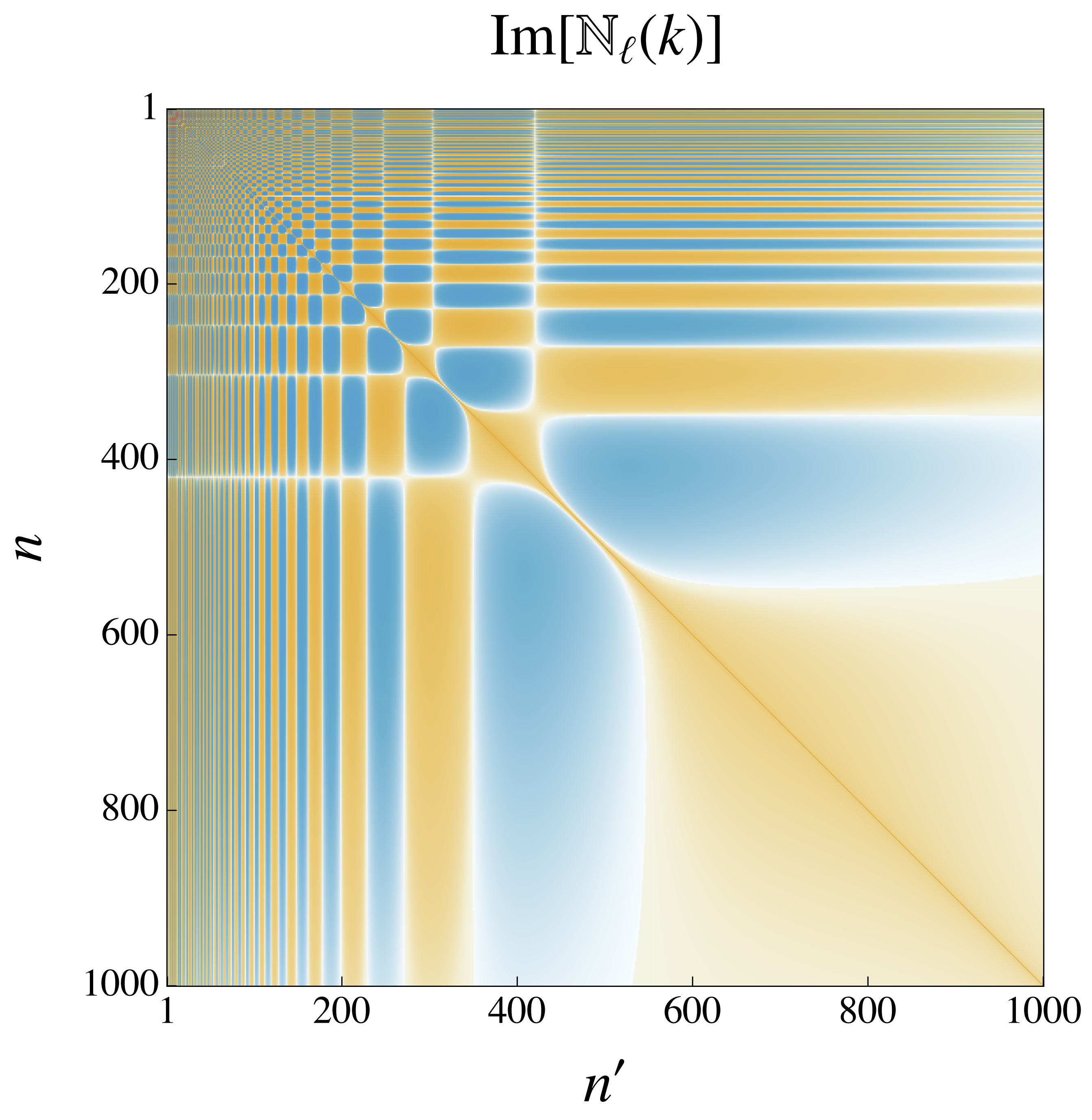

[Credit: NASA, FermiLAT]

Final thoughts

- We've developed a unitarization scheme which relies on an **anti-Hermitian** potential which is non-local and separable.
  - This potential ***is not*** a model of inelastic interactions but a natural consequence integrating out inelastic channels (**Feshbach projection**), the **optical theorem** and the **probability continuity equation**
- Our formalism has the ability to accommodate
  - **Bound state formation**
  - **Contact-type interactions**
- Future directions:
  - Exploration of the implications for self-interacting dark matter
- **Implications**:
  - Dark matter production
  - Self-interactions and galactic structure
  - Indirect detection experiments

[**MMF**, Petraki: 2405.02222, **2512.02097**, 2602.20243]





Extra Slides

Unitarity review

$$\underline{\text{“probabilities sum to 1”}}$$

$$\underbrace{\mathbb{1} = SS^\dagger = S^\dagger S, \qquad S = \mathbb{1} + \mathbb{i}T}$$

$$-\mathbb{i}(T - T^\dagger) = TT^\dagger = T^\dagger T.$$

$$\Downarrow$$

$$-\mathbb{i}(\mathcal{M}^{ab} - \mathcal{M}^{ba*}) = \sum_c \int \mathrm{d}\Pi_c\, \mathcal{M}^{ac}\mathcal{M}^{bc*}(2\pi)^4\delta^4(\{k^a\} - \{k^c\})$$

$$\Downarrow\ \sigma \propto |\mathcal{M}|^2$$

$$\sigma_\ell^{\mathrm{inel}}/\sigma_\ell^U \leq \sqrt{\sigma_\ell^{\mathrm{elas}}/\sigma_\ell^U}\left(1 - \sqrt{\sigma_\ell^{\mathrm{elas}}/\sigma_\ell^U}\right) \quad \text{with} \quad \sigma_\ell^U = \frac{2^\delta 4\pi(2\ell+1)}{\mathbf{k}^2}$$

General Regularization Matrix

- We can define the following quantities

$$w^j_{A,\ell} \equiv \left|\left[\eta_{A,\ell}\,\eta_\ell^{-1}\,(\mathbb{1} + \eta_\ell\,\mathbb{W}_\ell\,\eta_\ell)^{-1}\,\eta_\ell \hat{M}_{\ell,\mathrm{unreg}}\right]^j\right|^2 ,$$

$$w_\ell \equiv -\frac{\mathbb{i}}{2}\hat{M}^\dagger_{\ell,\mathrm{unreg}}\left[\eta_\ell(\mathbb{1} + \eta_\ell\,\mathbb{W}_\ell\,\eta_\ell)^{-1}\eta_\ell - \mathrm{h.c.}\right]\hat{M}_{\ell,\mathrm{unreg}},$$

$$\tilde{w}_\ell \equiv +\frac{1}{2}\hat{M}^\dagger_{\ell,\mathrm{unreg}}\left[\eta_\ell(\mathbb{1} + \eta_\ell\,\mathbb{W}_\ell\,\eta_\ell)^{-1}\eta_\ell + \mathrm{h.c.}\right]\hat{M}_{\ell,\mathrm{unreg}},$$

- Which allows us to obtain

$$x_{\ell,\mathrm{reg}} = \frac{x_{\ell,\mathrm{unreg}} + (1 - x_{\ell,\mathrm{unreg}})(w_\ell^2 + \tilde{w}_\ell^2) \pm 2\tilde{w}_\ell\sqrt{x_{\ell,\mathrm{unreg}}(1 - x_{\ell,\mathrm{unreg}})}}{(1 + w_\ell)^2 + \tilde{w}_\ell^2},$$

$$y^j_{\ell,\mathrm{reg}} = \frac{w^j_{A,\ell}}{(1 + w_\ell)^2 + \tilde{w}_\ell^2},$$

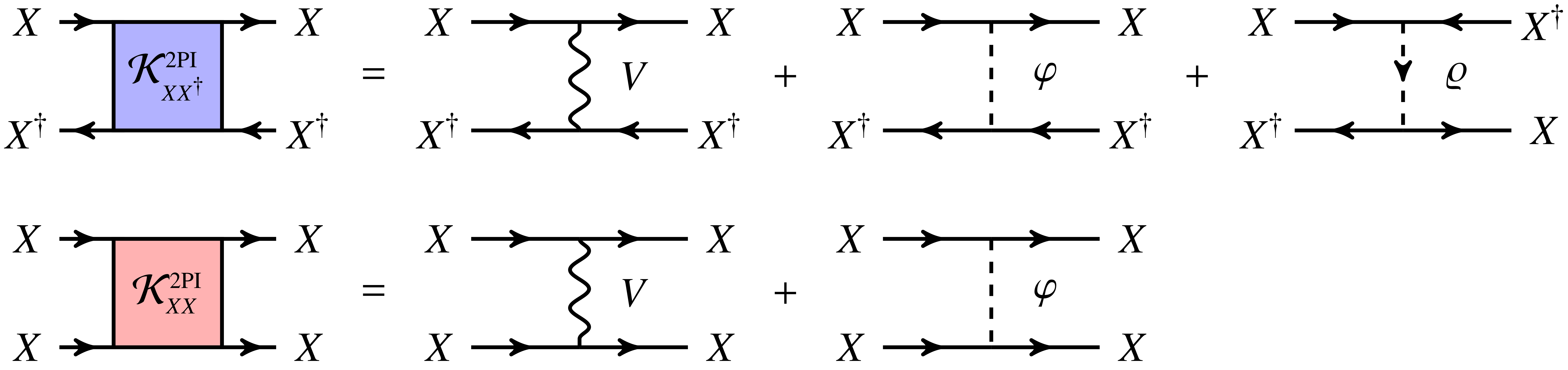
$$y_{\ell,\mathrm{reg}} = \frac{w_\ell}{(1 + w_\ell)^2 + \tilde{w}_\ell^2}.$$

Unitarity and bound-state formation: Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu X)^\dagger(D^\mu X) + (D_\mu \varrho)^\dagger(D^\mu \varrho) + \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - m_X^2|X|^2 - m_\varrho^2|\varrho|^2 - \frac{1}{2}m_\varphi^2\varphi^2$$
$$- \boxed{\frac{y_{X\varrho}\, m_X}{2}\left(X^2\varrho^\dagger + X^{\dagger\,2}\varrho\right)} - y_{X\varphi}\, m_X\, |X|^2\varphi - (y_{\varrho\varphi}\, m_\varphi)\, |\varrho|^2\varphi - \frac{y_\varphi\, m_\varphi}{3}\,\varphi^3$$
$$- \frac{\lambda_X}{4}|X|^4 - \frac{\lambda_\varrho}{4}|\varrho|^4 - \frac{\lambda_\varphi}{4!}\varphi^4 - \lambda_{X\varrho}|X|^2|\varrho|^2 - \frac{\lambda_{X\varphi}}{2}|X|^2\varphi^2 - \frac{\lambda_{\varrho\varphi}}{2}|\varrho|^2\varphi^2$$

$$\alpha_V \equiv g^2/(4\pi), \quad \alpha_\varrho \equiv y_{X\varrho}^2/(16\pi), \quad \alpha_\varphi \equiv y_{X\varphi}^2/(16\pi)$$

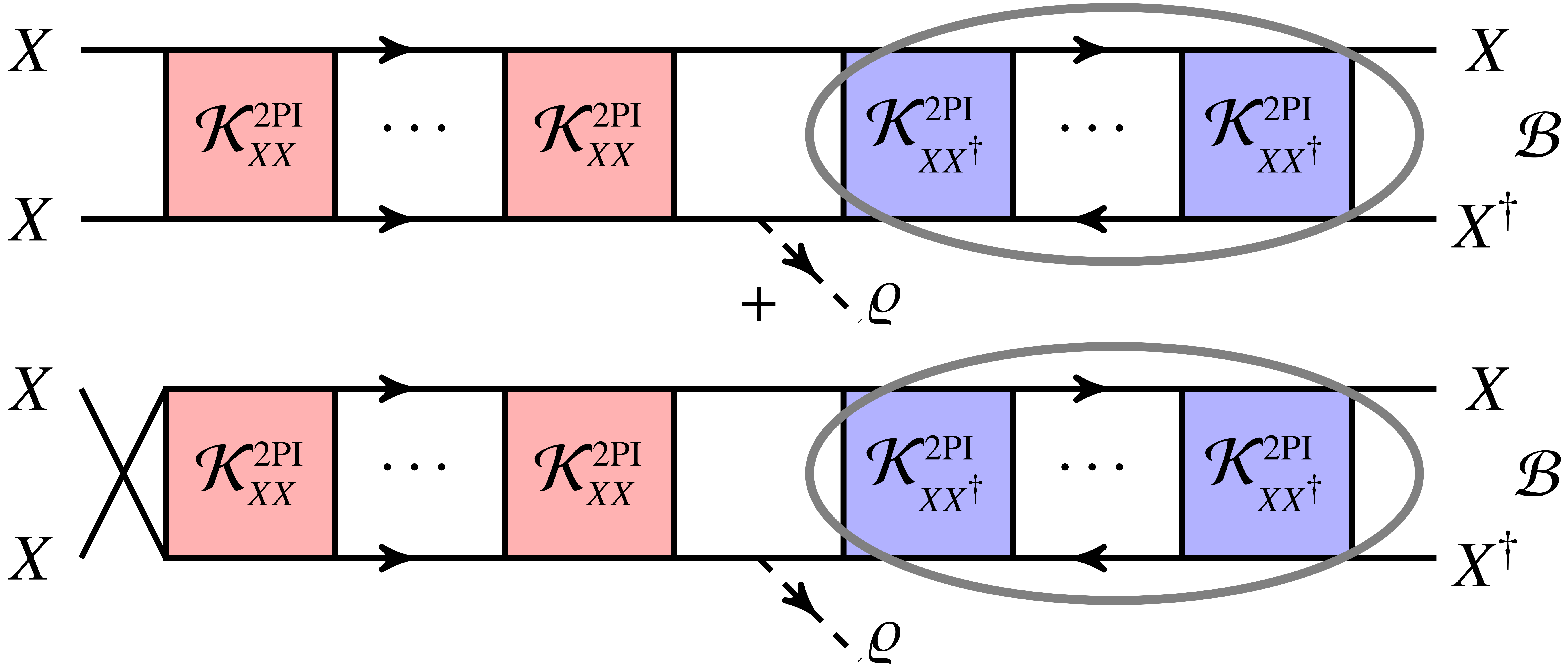
Unitarity and bound-state formation: Potentials



$$V_{XX^\dagger}(r) = -\frac{1}{r}\left(+\alpha_V + \alpha_\varphi e^{-m_\varphi r} + (-1)^\ell \alpha_\varrho e^{-m_\varrho r}\right),$$

$$V_{XX}(r) = V_{X^\dagger X^\dagger}(r) = -\frac{1+(-1)^\ell}{2}\frac{1}{r}\left(-\alpha_V + \alpha_\varphi e^{-m_\varphi r}\right),$$

Unitarity and bound-state formation: Potentials



General Regularization Matrix

- An alternate form:

$$[\mathbb{N}_\ell(k)]^{ij} \equiv \delta^{ij} - \eta^i_\ell \eta^j_\ell \int_0^\infty \mathrm{d}r\, r \int_0^\infty \mathrm{d}r'\, r' \left[v^i_\ell(r)\, G_{k,\ell}(r,r')\, v^{j*}_\ell(r') \right]$$

- With the Green's function:

$$G_{k,\ell}(r,r') = + \frac{2\mu i}{\vartheta_\ell k} \mathcal{F}^*_{k,\ell}(r_<) \mathcal{H}^{(+)}_{k,\ell}(r_>)$$

Proven

w/ Jost function formalism

$$= \frac{2\mu}{\vartheta \ell} \left[\frac{1}{\pi} \int_{-\infty}^\infty \mathrm{d}q\, \frac{\mathcal{F}_{q,\ell}(r)\, \mathcal{F}^*_{q,\ell}(r')}{q^2 - k^2 - i\epsilon} - \sum_n \frac{\mathcal{B}_{n\ell}(r) \mathcal{B}^*_{n\ell}(r')}{\kappa^2_{n\ell} + k^2} \right]$$