Casimir energy in non abelian gauge theories

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Gauge theories

- Gauge theories are the basis of the interaction theory on the Standard Model of particle physics
- The self interacting nature of gauge fields makes the analytic treatment of these theories very challenging on the non-abelian case
- In the low energy regime the perturbative methods fail to explain phenomena like quark confinement and the existence of a mass gap
- Evidences of the existence of this confinement mechanism and mass gap in the glueball spectra have been obtained using numerical simulations
- Glueball: massive particles formed by gluons
- Apart from these numerical results very little is known of the vacuum structure of gauge theories
- Confine the field in a domain where the boundary can have different types of boundary conditions $\rightarrow$ Casimir effect

Casimir effect

- A small force that appears in the vacuum of quantum field theories when boundary conditions are imposed
- First proposed by Hendrik Casimir in 1948
- One of the first applications of boundary effects in QFT
- Has been measured experimentally in different set ups for electromagnetic case
- Perturbative calculations of the Casimir energy on non-abelian gauge theories have been done but give a massless decay(potential)
- Some progress has been achieved in the analytic approach to gauge theories developed by D. Karabali and V. P. Nair in 2+1 dimensions the gauge invariant degrees of freedom are parametrized by some approximations with scalar fields and mass given by the anomalies $m = \frac{g^2 c_a}{2\pi}$

Goals

- The main goal is to obtain the Casimir energy of non-abelian gauge theories by non-perturbative Monte Carlo simulations in 2+1 and 3+1 dimensions
- Analyze its dependence focusing on the decay: potential (massless) or exponential (massive) decay with the distance between the boundaries
- In the latter we will compute the mass that drives the exponential decay and compare it with the glueball masses
- In 2+1 dimension it will allow us to compare with the expected mass of the Karabali-Nahir parametrization

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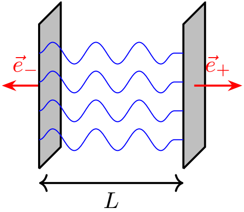
Boundary conditions

- We consider a massive free scalar field in D+1 dimensions confined between homogeneous infinite walls
- The general boundary conditions are given by 2x2 unitary matrices



$$\begin{pmatrix}\varphi(L/2)\\ \varphi(-L/2)\end{pmatrix} - i\delta\begin{pmatrix}\dot{\varphi}(L/2)\\ \dot{\varphi}(-L/2)\end{pmatrix} = U\left(\begin{pmatrix}\varphi(L/2)\\ \varphi(-L/2)\end{pmatrix} + i\delta\begin{pmatrix}\dot{\varphi}(L/2)\\ \dot{\varphi}(-L/2)\end{pmatrix}\right)$$

Which we can parametrize as [1]

$$U(\alpha,\gamma,\mathrm{n}) = \mathrm{e}^{i\alpha}\left(\mathbb{I}\cos\gamma + i\mathrm{n}\cdot\boldsymbol{\sigma}\,\sin\gamma\right); \quad \alpha\in[0,2\pi],\ \gamma\in[-\pi/2,\pi/2]$$

[1] M. Asorey, A. Ibort, and G. Marmo. "Global theory of quantum boundary conditions and topology change". In: International Journal of Modern Physics A 20.05 (2005), pp. 1001–1025.

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Vacuum Energy

- The vacuum energy with this set-up has the following divergences

$$E_0 = C_0(m)AL + C_1(m)A + C_c(m, L)$$

- $C_0(m)$ is the bulk divergence, $C_1(m)$ is the divergence of the boundary walls and $C_c(m)$ is the Casimir energy

$$E_U(L, m) = \lim_{L_0 \to \infty} (E_0(L, m) + E_0(2L_0 + L, m) - 2E_0(L_0 + L, m))$$

- Cancels every term that does not go to zero as L grows which is the Casimir energy

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Casimir Energy

- Applying this renormalization prescription we obtain the Casimir energy for general boundary conditions

$$E_U(L,m) = \frac{A}{2^{D+1}\pi^{D/2}\Gamma\left(\frac{D}{2}+1\right)} \int_m^\infty dk\, \left(k^2 - m^2\right)^{D/2} \left(L - \frac{d}{dk}\log\frac{h_U^L(ik)}{h_U^\infty(ik)}\right),$$

- with h_U^L the spectral function

$$h_U^L(k) = \sin(kL)\left((k^2-1)\cos\gamma + (k^2+1)\cos\alpha\right) - 2k\sin\alpha\cos(kL) - 2kn_1\sin\gamma$$

- The asymptotic behaviour with mL classifies the boundary conditions in two families

$$E_U \sim \begin{cases} e^{-mL} & \text{if } \operatorname{tr}(U\sigma_1) \neq 0 \rightarrow \text{related constrains between boundaries} \\ e^{-2mL} & \text{if } \operatorname{tr}(U\sigma_1) = 0 \rightarrow \text{independent constrains between boundaries,} \end{cases}$$

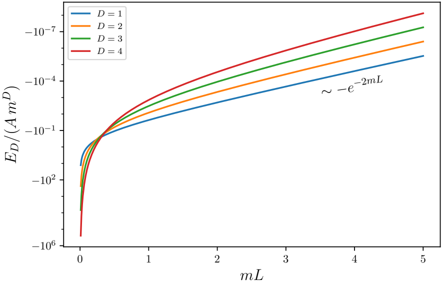
Dirichlet/Neumann boundary conditions

- $\varphi(L/2) = \varphi(-L/2) = 0$ / $\dot{\varphi}(L/2) = \dot{\varphi}(-L/2) = 0$
- The Casimir energy is given by

$$E_D(L, m) = -\frac{A m^{\frac{D+1}{2}}}{2^D \pi^{\frac{D+1}{2}} L^{\frac{D-1}{2}}} \sum_{j=1}^{\infty} \frac{1}{j^{\frac{D+1}{2}}} K_{\frac{D+1}{2}}(2jmL).$$

- It shows an asymptotic behaviour with e^{-2mL}

$$E_D(L, m) = -\frac{A m^{D/2}}{2^{D+1} \pi^{D/2} L^{D/2}} e^{-2mL} (1 + O(1/mL)) + O(e^{-4mL}).$$



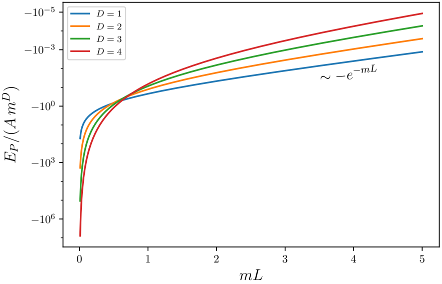
Periodic boundary conditions

- $\varphi(L/2) = \varphi(-L/2)$ and $\dot{\varphi}(L/2) = -\dot{\varphi}(-L/2)$
- The Casimir energy is given by

$$E_P(L,m) = -\frac{Am^{\frac{D+1}{2}}}{2^{\frac{D-1}{2}}\pi^{\frac{D+1}{2}}L^{\frac{D-1}{2}}}\sum_{j=1}^{\infty}\frac{1}{j^{\frac{D+1}{2}}}K_{\frac{D+1}{2}}(jmL),$$

- It shows an asymptotic behaviour with e^{-mL}

$$E_P(L,m) = -\frac{Am^{D/2}}{(2\pi L)^{D/2}}e^{-mL}\left(1+O(1/mL)\right)+O(e^{-2mL}).$$



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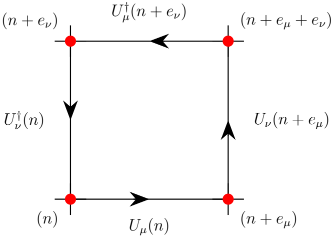
Wilson action

- The Wilson action for a pure Yang-Mills $SU(N_c)$ theory is

$$S_W = \beta \sum_{n\in\Lambda} \sum_{\mu<\nu} \left(1 - \frac{1}{N_c} \mathrm{Re}\ \mathrm{tr}\ P_{\mu\nu}(n)\right)$$

with

$$P_{\mu\nu}(n) = U_\mu(n) U_\nu(n+e_\mu) U^\dagger_\mu(n+e_\nu) U^\dagger_\nu(n)$$



- $U_\mu(x)$ are the link variables between the lattice sites that are $SU(n)$ matrices. This link variables transforms under gauge transformation as

$$U_\mu(x) \to \Omega(x) U_\mu(x) \Omega^\dagger(x+ae_\mu),$$

- In SU(2) the link variables are gonna be 2x2 matrices and $\beta = \frac{4}{a^{3-D} g^2}$

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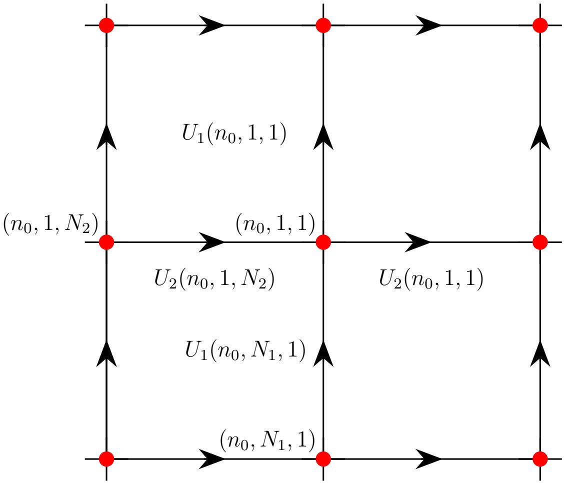
Periodic boundary conditions

- In pbc the last link variable connects the boundary points in the lattice
- We apply this bc always to all directions but the transverse one



Perfect colour conductor boundary conditions

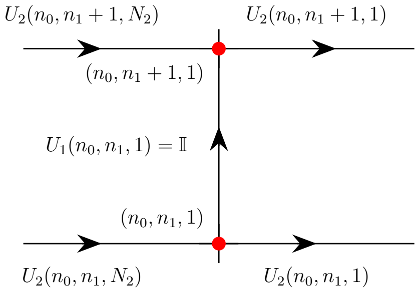
- The equivalent condition to the perfect conductor in EM
  $E_1 = E_2 = B_3 = 0 \Rightarrow F_{01} = F_{02} = F_{12} = 0$
- In the lattice it becomes $P_{01} = P_{02} = P_{12} = \mathbb{I}$
- We fix the links in the boundary to the identity

$$U_0(n_0, n_1, n_2, 1) = U_1(n_0, n_1, n_2, 1) = U_2(n_0, n_1, n_2, 1) = \mathbb{I}$$

and apply pbc on the on the other side.

- In the 2+1 case we have
  $F_{01} = 0 \Rightarrow P_{01} = \mathbb{I}$

$$U_0(n_0, n_1, 1) = U_1(n_0, n_1, 1) = \mathbb{I}$$



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String tension

- To give our results for the masses of the expected exponential decay we use the string tension $\sqrt{\sigma}$ (energy units)
- Is the coefficient of the linear term in the static quark-antiquark potential
- It gives the decay of large Wilson loops

$$\ln W = -(a^2\sigma)A + O(P)$$

- The relevant masses are
  - lightest glueball in 2+1: $4.7\sqrt{\sigma}$ [2]
  - lightest glueball in 3+1: $3.78\sqrt{\sigma}$ [3]
  - Karabali-Nahir mass in 2+1: $0.92\sqrt{\sigma}$ [4]

[2] A. Athenodorou and M. Teper. "SU(N) gauge theories in 2+ 1 dimensions: glueball spectra and k-string tensions". In: Journal of High Energy Physics 2017.2 (2017), pp. 1–55.

[3] A. Athenodorou and M. Teper. "SU (N) gauge theories in 3+ 1 dimensions: glueball spectrum, string tensions and topology". In: Journal of High Energy Physics 2021.12 (2021), pp. 1–119.

[4] D. Karabali and V. P. Nair. "Casimir effect in (2+ 1)-dimensional Yang-Mills theory as a probe of the magnetic mass". In: Physical Review D 98.10 (2018), p. 105009.

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Inner energy

$$\mathcal{E} = -\frac{\beta}{aN_0} \sum_{n \in \Lambda} \left(1 + \frac{1}{2} \mathrm{tr} \left(P_{12}(n) - P_{01}(n) - P_{02}(n) \right) \right)$$

- That we can simplify by fixing $N_0 = N_1$ to

$$\langle \mathcal{E}(N_A, N_L, \beta) \rangle = -\frac{u^4 \beta}{aN_A} \left(1 - \frac{1}{2} \langle \sum_{n \in \Lambda} \mathrm{tr} \left(P_{01}(n) \right) \rangle \right).$$

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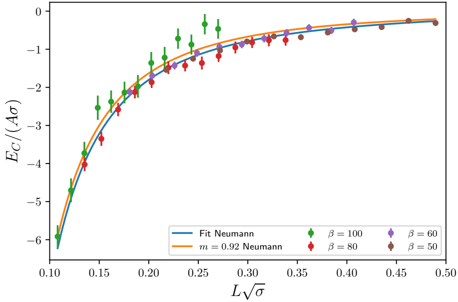
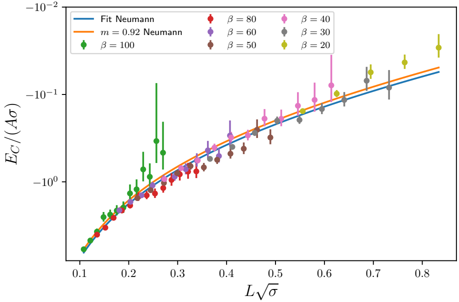
Casimir energy perfect colour conductor

- We can compute the Casimir energy as

$$E_C(N_L,\beta) = \lim_{N_{L_0}\to\infty}\left(\langle\mathcal{E}_C(N_L,\beta)\rangle - \langle\mathcal{E}_C(N_{L_0},\beta)\rangle - \frac{N_L - N_{L_0}}{N_{L_0}}\langle\mathcal{E}_P(N_{L_0},\beta)\rangle\right)$$

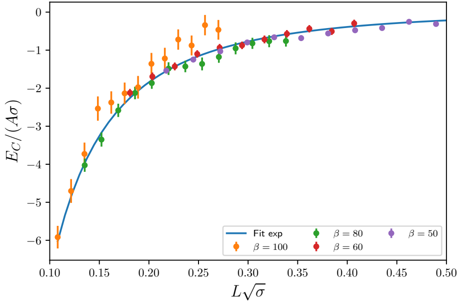


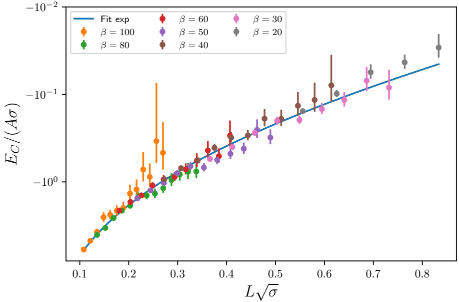


$$\frac{E_C(L\sqrt{\sigma})}{A\sigma} = -\frac{G}{16\pi L^2\sigma}\left(2mL\,\mathrm{Li}_2\left(e^{-2mL}\right) + \mathrm{Li}_3\left(e^{-2mL}\right)\right)$$

- Best fit $G = 3.26 \pm 0.06$ and $\frac{m}{\sqrt{\sigma}} = 0.89 \pm 0.07$. $(m_{\text{glueball}} = 4.7\sqrt{\sigma})$

Casimir energy perfect colour conductor





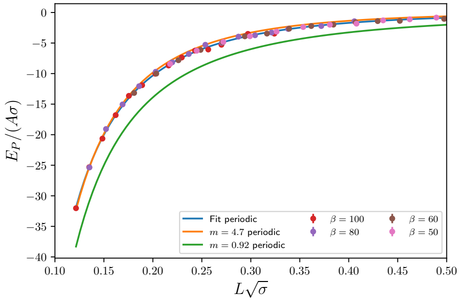
$$\frac{E_C(L\sqrt{\sigma})}{A\sigma} = -\frac{C}{(L\sqrt{\sigma})^{\nu}} e^{-2\mu L}$$

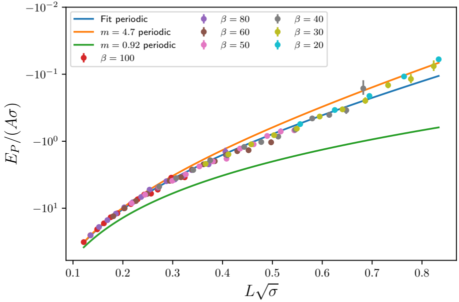
- Best fit parameters $C = 0.23 \pm 0.04$, $\frac{\mu}{\sqrt{\sigma}} = 1.14 \pm 0.15$ and $\nu = 1.58 \pm 0.07$,
- Both fits give masses much smaller then the glueball $m_g = 4.7\sqrt{\sigma}$
- Also compatible with being parametrized by 3 scalar fields with $m = 0.92\sqrt{\sigma}$.

Casimir energy periodic

- We can compute the Casimir energy with periodic boundary conditions as

$$E_P(N_L,\beta) = \lim_{N_{L_P}\to\infty}\left(\langle \mathcal{E}_P(N_L,\beta)\rangle - \frac{N_L}{N_{L_P}}\langle \mathcal{E}_P(N_{L_P},\beta)\rangle\right)$$

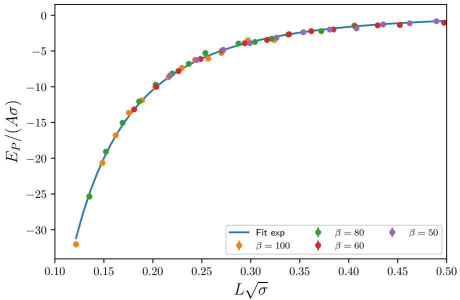


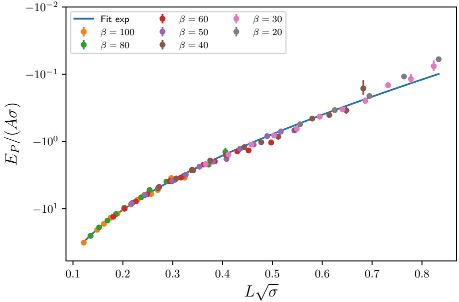


$$\frac{E_P(L\sqrt{\sigma})}{A\sigma} = -\frac{G}{2\pi L^2\sigma}\left(mL\,\mathrm{Li}_2\left(e^{-mL}\right) + \mathrm{Li}_3\left(e^{-mL}\right)\right)$$

- Best fit $G = 2.90 \pm 0.02$ and $\frac{m}{\sqrt{\sigma}} = 3.96 \pm 0.03$. $(m_{\text{glueball}} = 4.7\sqrt{\sigma})$

Casimir energy periodic





$$\frac{E_P(L\sqrt{\sigma})}{A\sigma} = -\frac{C}{L(\sqrt{\sigma})^{\nu}} e^{-\mu L}$$

- Best fit parameters $C = 1.64 \pm 0.09$, $\frac{\mu}{\sqrt{\sigma}} = 3.72 \pm 0.08$ and $\nu = 1.61 \pm 0.02$
- Both masses are closer to the glueball ($m_g = 4.7\sqrt{\sigma}$) but still lower
- Much higher value than the pccbc $m_{pcc} = 0.89\sqrt{\sigma}$.

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$$\mathcal{E} = -\frac{\beta}{aN_0}\sum_{n\in\Lambda}\left(\frac{1}{2}\mathrm{tr}\left(P_{12}(n)+P_{23}(n)+P_{13}(n)-P_{01}(n)-P_{02}(n)-P_{03}(n)\right)\right)$$

- That we can simplify by fixing $N_0 = N_1 = N_2$ to

$$\langle\mathcal{E}(N_A, N_L, \beta)\rangle = -\frac{u^4\beta}{2aN_0}\,\langle\sum_{n\in\Lambda}\mathrm{tr}\left(\frac{P_{03}+P_{23}+P_{13}}{3}-\frac{P_{01}+P_{02}+P_{12}}{3}\right)\rangle\,.$$

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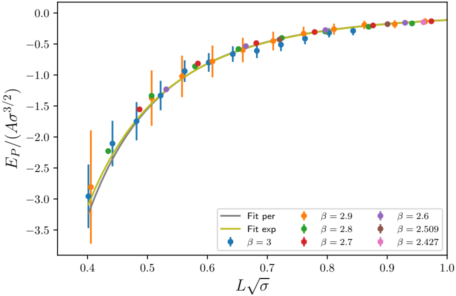
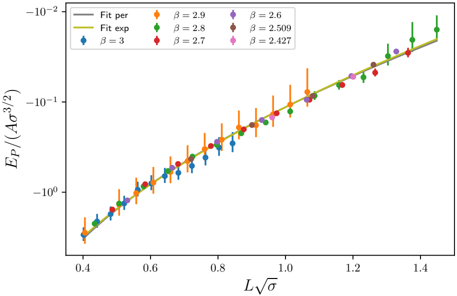
Casimir energy periodic

- We can compute the Casimir energy with periodic boundary conditions directly





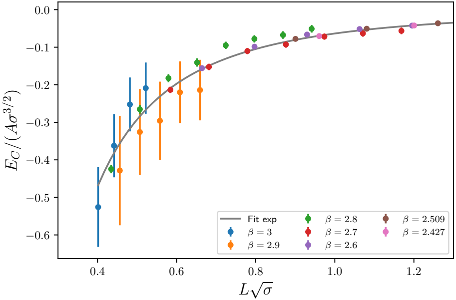
$$\frac{E_P(L\sqrt{\sigma})}{A\sigma^{3/2}} = -\frac{C}{(\sqrt{\sigma}L)^{\nu}} e^{-\mu L}$$

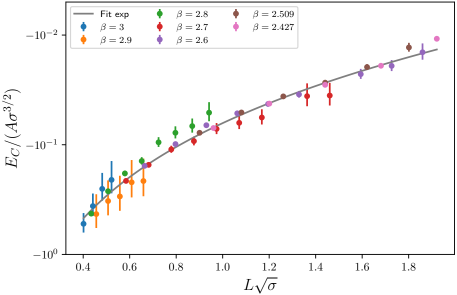
- Best fit $C = 0.79 \pm 0.02$, $\frac{\mu}{\sqrt{\sigma}} = 1.94 \pm 0.02$ and $\nu = 2.34 \pm 0.02$ $(m_{\text{glueball}} = 3.78\sqrt{\sigma})$

Casimir energy perfect conductor

- We can compute the Casimir energy with perfect colour conductor as

$$E_C(N_L,\beta) = \lim_{N_{L_0}\to\infty} \langle \mathcal{E}_C(N_L,\beta)\rangle - \langle \mathcal{E}_C(N_{L_0},\beta)\rangle$$





$$\frac{E_C(L\sqrt{\sigma})}{A\sigma^{3/2}} = -\frac{C}{(\sqrt{\sigma}L)^{\nu}} e^{-\mu L}$$

- Best fit: $C = 0.081 \pm 0.003$, $\frac{\mu}{\sqrt{\sigma}} = 0.24 \pm 0.04$ and $\nu = 2.03 \pm 0.04$. ($m_{\text{glueball}} = 3.78\sqrt{\sigma}$)

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Conclusions

- The ratio of the exponential decay of the Casimir energy in massive scalar fields classifies the boundary conditions in two different families
- The Casimir energy in non-abelian gauge theories presents an exponential decay characteristic of massive theories

	Glueball	Casimir Energy	Glueball/Casimir Energy
2+1	$m_g = 4.7\sqrt{\sigma}$	$m_p = 3.96\sqrt{\sigma}$	$m_p/m_g = 0.84$
		$m_{pcc} = 0.89\sqrt{\sigma}$	$m_{pcc}/m_g = 0.19$
3+1	$m_g = 3.78\sqrt{\sigma}$	$m_p = 1.94\sqrt{\sigma}$	$m_p/m_g = 0.51$
		$m_{pcc} = 0.24\sqrt{\sigma}$	$m_{pcc}/m_g = 0.06$

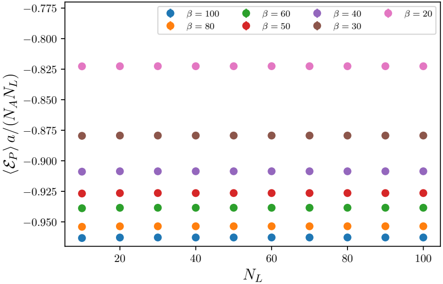
- Simulations with $SU(3)$ and other bc could clarify further this issues

Thanks for your attention!

Back up slides

Energy renormalization periodic 2+1

- We expect the boundary term $C_1(\beta)$ to cancel with pbc

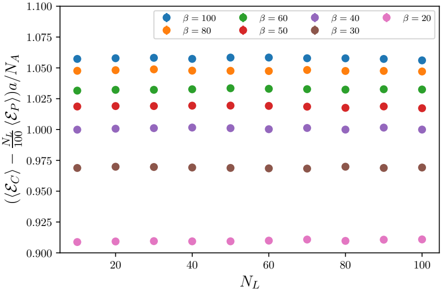


- We see there is only the $C_0(\beta)$ contribution, which means we can obtain the value as

$$C_0(\beta) = \lim_{N_{L_0} \to \infty} \langle \mathcal{E}_P(N_{L_0}, \beta) \rangle / (N_A N_{L_0})$$

Energy renormalization perfect colour conductor 2+1

- Once we subtract the term $C_0(\beta)$ we should observe just the term with $C_1(\beta)$

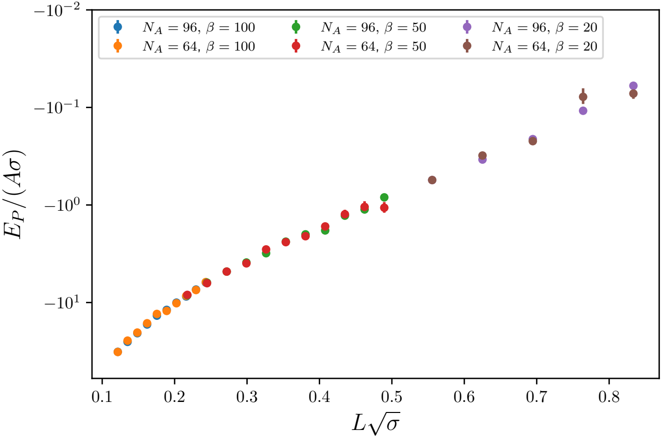


- We can see there is only left the $C_1(\beta)$ contribution, which means we can obtain the value as

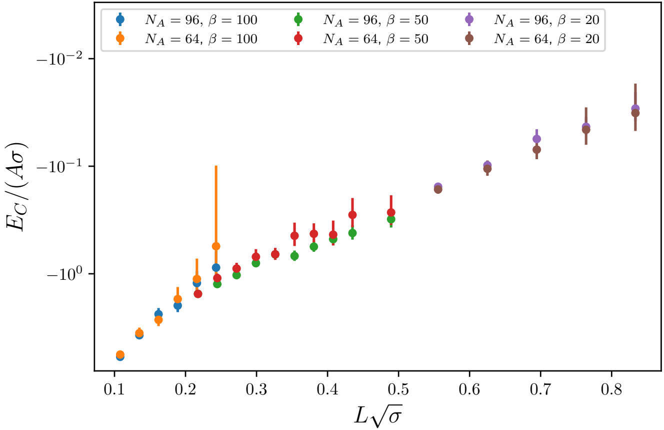
$$C_1(N_{L_0}, \beta) = \lim_{N_{L_0} \to \infty} \left(\langle \mathcal{E}_C(N_{L_0}, \beta) \rangle - \langle \mathcal{E}_P(N_{L_0}, \beta) \rangle \right) / N_A$$

Lattice volume error N_A periodic 2+1

- Different $N_A = N_0 = N_1$.

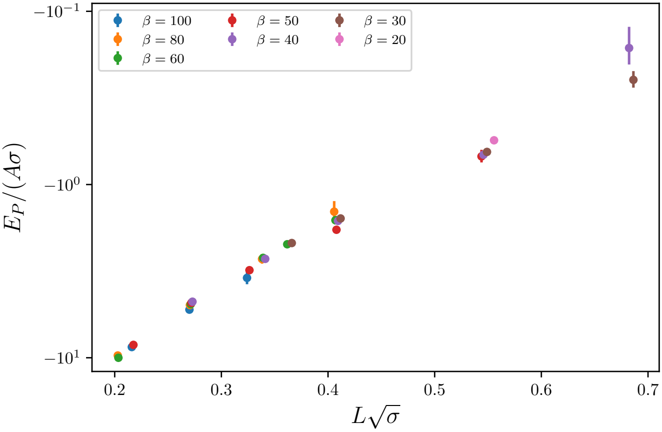


- Different $N_A = N_0 = N_1$.



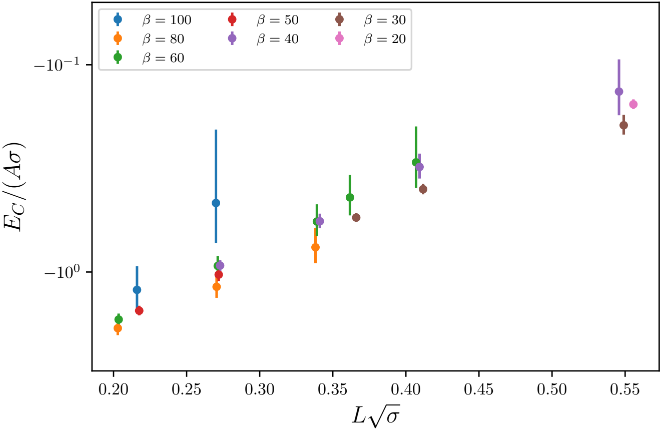
Lattice volume error N_2 periodic 2+1

- Different N_2.



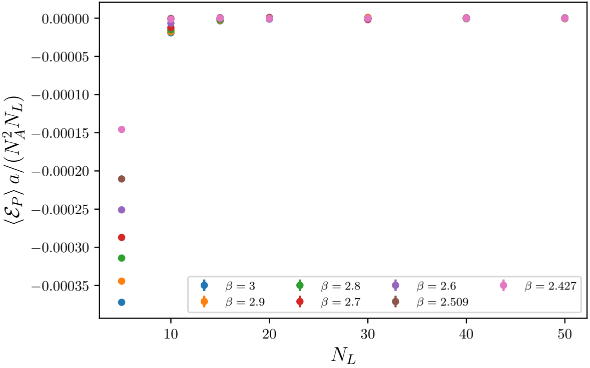
Lattice volume error N_2 perfect colour conductor 2+1

- Different N_2.



Energy renormalization periodic 3+1

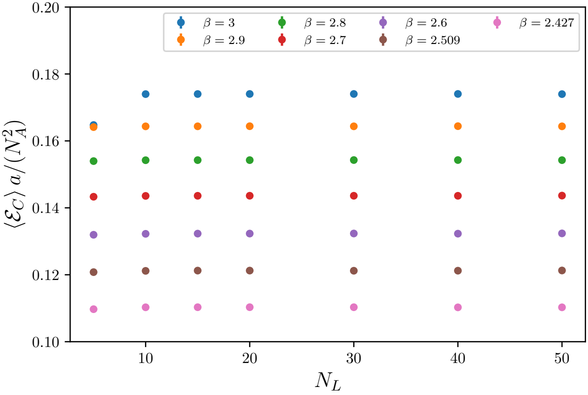
- The bulk contribution cancels due to the form of the energy-momentum tensor



- It gives directly the Casimir energy

Energy renormalization perfect colour conductor 3+1

- We should directly observe the boundary terms

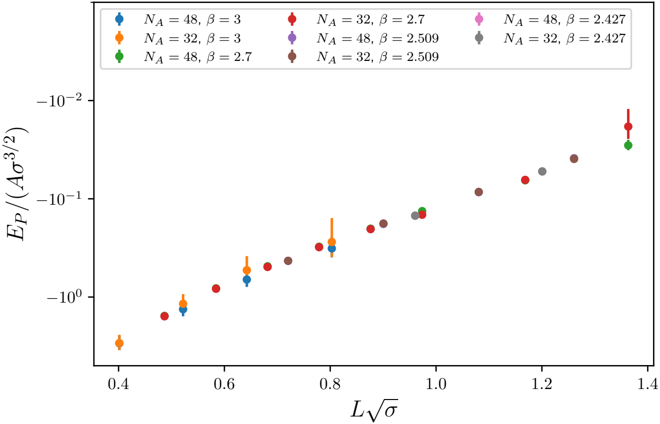


- We can compute it as

$$C_1(\beta) = \lim_{N_{L_0} \to \infty} \langle \mathcal{E}_C(N_{L_0}, \beta) \rangle / N_A^2$$

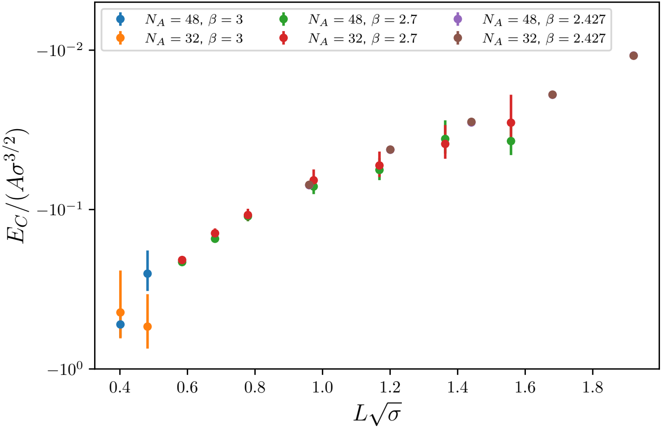
Lattice volume error N_A periodic 3+1

- Different $N_A = N_0 = N_1 = N_2$.



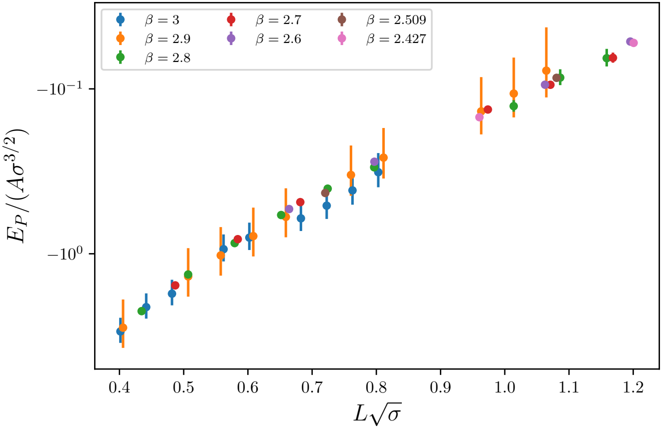
Lattice volume error N_A perfect colour conductor 3+1

- Different $N_A = N_0 = N_1 = N_2$.



Lattice volume error N_3 periodic 3+1

- Different N_3.



Lattice volume error N_3 perfect colour conductor 3+1

- Different N_3.